As per the Syllabus of WBUT

Second Edition

Basic Engineering Physics

Sujay Kumar Bhattacharya • Saumen Pal

For Subject Code: [illegible]

- Roadmap to the Syllabus
- [illegible] – [illegible] Solved WBUT Question Papers
- Model Question Papers

Basic Engineering Physics

Second Edition

WBUT-2011-12

Sujay Kumar Bhattacharya
Head of the Department
Science & Humanities
West Bengal Survey Institute
Bandel, Hooghly

Saumen Pal
Assistant Professor (Physics)
Department of Basic Sciences and Humanities
Meghnad Saha Institute of Technology
Kolkata

Tata McGraw Hill Education Private Limited
NEW DELHI

McGraw-Hill Offices

New Delhi New York St Louis San Francisco Auckland Bogotá Caracas
Kuala Lumpur Lisbon London Madrid Mexico City Milan Montreal
San Juan Santiago Singapore Sydney Tokyo Toronto

Contents

Roadmap to the Syllabus

Basic Engineering Physics (PHY 101/201)

Module 1: Oscillation

1.1 Simple harmonic motion, Preliminary concepts, Superposition of SHMs in two mutually perpendicular direction, Lissajous figure

1.2 Damped vibration, Differential equation and its solution, Critical damping, Logarithmic decrement, Quality factor

1.3 Forced vibration, Differential equation and its solution, Amplitude and Velocity resonance, Sharpness of resonance, Application L-C-R Circuit

GO TO
CHAPTER 1 SIMPLE HARMONIC MOTION
CHAPTER 2 FREE AND DAMPED VIBRATIONS
CHAPTER 3 FORCED VIBRATIONS

Module 2: Optics 1

2.1 Interference of electromagnetic waves, Conditions for sustained interference, double slit as an example, Qualitative idea of Spatial and Temporal Coherence, Conservation of energy and intensity distribution, Newton's ring (no deduction necessary)

2.2 Diffraction of light–Fresnel and Fraunhofer class, Fraunhofer diffraction for single slit and double slits. Intensity distribution of N-slits and plane transmission grating (no deduction of the intensity distributions is necessary), Missing orders, Rayleigh criterion, Resolving power of grating and microscope. (definition and formulae)

GO TO
CHAPTER 4 INTERFERENCE OF LIGHT
CHAPTER 5 DIFFRACTION OF LIGHT

Module 3: Optics 2

3.1 Polarization — General concept of polarization, plane of vibration and plane of polarization, Qualitative discussion on plane, Circulary and Elliptically polarized light, Polarization through reflection and Brewster's law, Double refraction (birefringence), Ordinary and Extraordinary rays, Nicol's Prism, Polaroid. Half wave plate and Quarter wave plate.

3.2 Lasar — Spontaneous and stimulated emission of radiation, population inversion, Einstein's A and B coefficients (derivation of the mutual relation), Optical resonator and condition necessary for active Laser action, Ruby Laser, He–Ne Laser— application of laser.

3.3 Holography — Theory of Holography, viewing the hologram, Applications.

GO TO
CHAPTER 6 POLARIZATION OF LIGHT
CHAPTER 7 LASER OPTICS
CHAPTER 8 HOLOGRAPHY

Module 4: Quantum Physics

4.1 Concept of dependence of mass with velocity, mass energy equivalence, energy- momentum relation (no deduction required). Black-body radiation— Rayleigh Jeans' law (derivation without the calculation of number of states)' wien's law, Ultraviolet catastrophy, planck's radiation law (Calculation of the average energy of the oscillator), Derivation of wien's displacement law and stephan's law from planck's radiation law. Raylegh jean's law and wien's law as limiting cases of planck's law, Compton effect (calculation of Compton wavelength is required).

4.2 Wave-particle duality and de Broglie's hypothesis, Concept of matter waves, Davisson-Germer experiment, Concept of wave packets and Heisenberg's uncertainty principle.

GO TO
CHAPTER 9 QUANTUM PHYSICS

Module 5: Crystallography

5.1 Elementary ideas of crystal structure — lattice, basis, unit cell, Fundamental types of lattices- Bravais lattice, Simple cubic, f.c.c. and b.c.c. lattices, (use of models in the class during teaching is desirable) Miller indices and miller planes, co-ordination number and Atomic packing factor.

5.2 X-rays — Origin of characteristic and Continuous X-ray, Bragg's law (No derivation), Determination of lattice constant.

GO TO
CHAPTER 10 CRYSTALLOGRAPHY

THE GREEK ALPHABET

Small	Capital	Name	Roman equivalent
α	A	alpha	a
β	B	beta	b
γ	E	gamma	g
δ	Γ	delta	d
ε	E	epsilon	e
ς	Z	zeta	z
η	H	eta	e
θ	Θ	theta	th
ι	I	iota	i
k	K	kappa	k
λ	Λ	lambda	l
μ	M	mu	m
ν	N	nu	n
ξ	Ξ	xi	x(ks)
o	O	omicron	o
π	Π	pi	p
ρ	ρ	rho	r
σ	Σ	sigma	s
τ	T	tau	t
υ	Ψ	upsilon	u
φ	Φ	phi	ph
χ	Ξ	chi	kh
ψ	Ψ	psi	ps
ω	Ω	omega	o

SOME USEFUL DERIVED UNITS

Physical Quantity	Derived unit	Symbol
Area	square meter	m^2
Volume	cubic meter or litre	m^3 or 1
Force	newton	N
Pressure and Stress	pascal	Pa (or N/m^2)
Work, Energy, Heat	joule	J
Power	watt	W
Moment of inertia	kgm^2	I
Torque	Nm	T
Specific heat capacity	$J/(kg\ k)$	c
Thermal conductivity	$W/(mk)$	k
Luminance	lux	lx
Luminous flux	lumen	lm

Electric charge	coulomb	C
Electric potential	volt	V
Electric resistance	ohm	Ω
Frequency	hertz	Hz

PREFIXES AS MULTIPLYING FACTORS

Prefix	Symbol	(Multiplying) factor value
atto –	a	10^{-18}
femto –	f	10^{-15}
pico –	p	10^{-12}
nano –	n	10^{-9}
micro –	μ	10^{-6}
milli –	m	10^{-3}
centi –	c	10^{-2}
deci –	d	10^{-1}
deca –	da	10^{1}
hecto –	h	10^{2}
kilo –	k	10^{3}
mega –	M	10^{6}
giga –	G	10^{9}
tera –	T	10^{12}
peta –	P	10^{15}
exa –	E	10^{18}

Examples:

$$1 \text{ fm} = 1 \times 10^{-15} \text{ m},$$
$$1 \text{ mg} = 1 \times 10^{-3} \text{ g},$$
$$1 \text{ M watt} = 1 \times 10^{6} \text{ watt etc.}$$

CHAPTER

1

Simple Harmonic Motion

1.1 INTRODUCTION

The simple harmonic motion is a special case of the generalized periodic motion. In our daily life we come across numerous things. We can classify them into two categories. The bodies which remain at rest with respect to us and the bodies which move with respect to us. The motions of all moving bodies can be categorized into two categories, namely, (i) the motion in which the body moves about a mean position (i.e., fixed point), and (ii) the motion in which the body moves from one place to another with respect to time. The first category of motion is called periodic motion while the second one is called translatory motion.

A moving bus, a flying aeroplane, a moving football, etc., are examples of translational motion while the motion of a simple pendulum, a spring-mass system, vibrations of a stretched string, etc., are examples of periodic motion.

Sometimes both the categories of motion can be observed in the same phenomenon depending on the observer's point of view. The waves in the sea appear to us to move towards the beach but the water moves up and down about a mean position. When someone displaces a stretched string, the displacement pulse travels from one end to the other end of the string but the material of the string vibrates about a mean position without getting itself translated forward. **The periodic motion in which the concerned body moves in a straight line is known as simple harmonic motion.**

1.2 RELATION OF SIMPLE HARMONIC MOTION WITH CIRCULAR MOTION

Simple harmonic motion (i.e., oscillation) is a special case of circular motion. To understand this relationship, let us discuss this with the help of a diagram. Figure 1.1 shows that *PAB* is a circle with its centre at *C*. Let us assume that *P* is a moving point on the circumference of the circle *PAB*. At time $t = 0$, it was at the point *O* and it has taken *t* units of time to reach the position *P*. So, the angle made by it with the line *CO* is $\theta = \omega t$ where ω is the uniform angular velocity of the particle. Let us drop a perpendicular *PM* from the point *P* to the diameter *YY′* of the circle. So, the point *M* also will be a moving point. While *P* will make an angle of 2π radian ($\theta = 360°$), the point *M* will make a complete oscillation. That is, when *P* will start moving from *O* towards the point *P*, *M* will start moving from *C* towards the point *M*. *M* will make a complete oscillation by moving from *C* to *Y*, then from *Y* to *Y′* and lastly from *Y′* to *C* during the time when *P* will make one complete

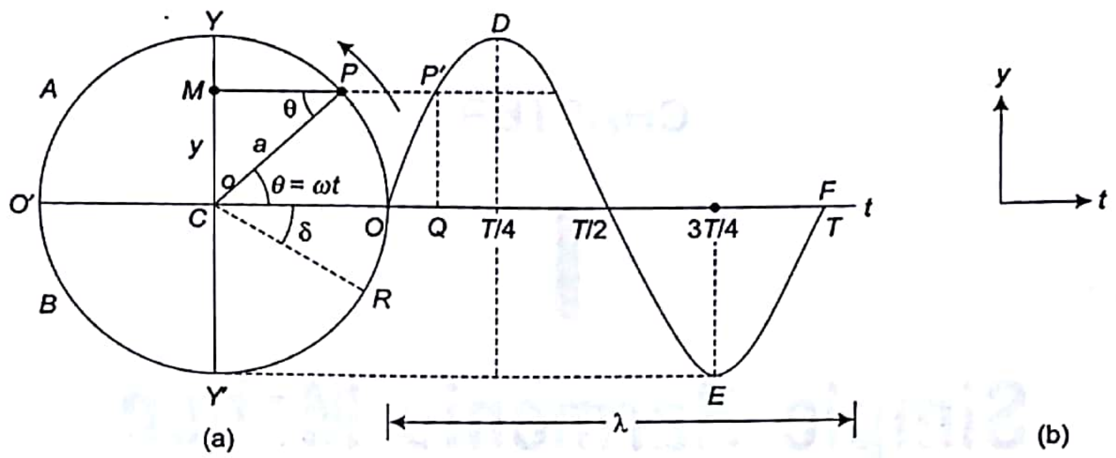


Fig. 1.1 Graph of simple harmonic oscillation along with its reference circle.

revolution on the circumference of the said circle by making an angular displacement of $\theta = 2\pi$. If one plots a graph of y against t, then one will get the graph $OP'DEF$. The circle along which the point P moves is called the reference circle. From Fig. 1.1 we can write,

$$\frac{MC}{CP} = \frac{y}{a} = \sin\theta$$

or, $$y = a\sin\theta \qquad \text{...(1.1)}$$

Equation (1.1) represents the oscillation (i.e., vibration) of the moving particle M on the straight path YCY' while the other moving particle P circulates on the circumference of the circle $OPABRO$.

Now if we consider M as our simple harmonic oscillator, we can consider P as the reference point which circulates uniformly on the reference circle with an angular velocity ω. If P remains at R instead of O, at $t = 0$, then the reference point P will start moving from the point R instead of the point O. In that case we can represent the motion of M by the following equation

$$y = a\sin(\theta + \delta) \qquad \text{...(1.2)}$$

So Eq. (1.2) represents the generalized equation of a simple harmonic oscillator. If now, we put $\theta = \omega t$, then we can write Eq. (1.2) as

$$y = a\sin(\omega t + \delta) \qquad \text{...(1.3)}$$

CM (= y) is called the displacement of the oscillator M, i.e., the vibrating particle. The displacement of a vibrating particle at any instant of time can be defined as its distance from the mean position of rest. In this case, C is the position of rest of the oscillating particle M. **The maximum displacement of a vibrating particle is called its amplitude of vibration.**

The instantaneous displacement y is given by,

$$y = a\sin(\theta + \delta)$$

or, $$y = a\sin(\omega t + \delta) \qquad \text{...(1.3)}$$

where δ is the initial phase of the oscillator and a is the amplitude

$\therefore$ $$y_{max} = a$$

The rate of change of displacement is called the velocity of the vibrating (i.e., oscillating) particle.

$\therefore$ velocity, $$v = \frac{dy}{dt} = a\,\omega\cos(\omega t + \delta) \qquad \text{...(1.4)}$$

The rate of change of velocity of an oscillating particle is called its acceleration.

∴ the acceleration is given by,

$$f = \frac{d}{dt}(v) = \frac{d^2}{dt^2}(y)$$
$$= -a\omega^2 \sin(\omega t + \delta)$$

or, $$f = -a\omega^2 \sin(\omega t + \delta) \quad \text{...(1.5)}$$

where f represents the acceleration of the oscillating particle.

Now using Eq. (1.3) we can write the Eq. (1.5) as follows:

$$f = -\omega^2 y \quad \text{...(1.6)}$$

At the extreme positions, where y is maximum, the velocity $v = \frac{dy}{dt} = 0$ and the acceleration $f = \frac{d^2y}{dt^2}$ is maximum and it is directed towards the mean position. The returning force induces a negative velocity at the point of return. When the displacement y becomes zero, the velocity v becomes maximum and when the velocity v becomes zero, the acceleration becomes maximum in magnitude. The returning force again induces a velocity in the opposite direction. It becomes maximum when the displacement again becomes zero. The particle overshots the mean position due to its velocity. The process repeats itself periodically. Thus, the system vibrates. And in this way, displacement y, velocity v and acceleration f continuously keep on changing with respect to time.

Thus, the velocity of the vibrating particle becomes maximum (either in the direction of CY or that of CY') at the mean position of rest and it becomes zero at the extreme positions of vibration. The acceleration of the particle becomes zero at the mean position of rest and it becomes maximum at the extreme positions of vibration. And the acceleration is always directed towards the mean position of rest and it is also directly proportional to the displacement of the vibrating particle. **So, we can define simple harmonic motion as follows: It is such a motion where the acceleration is always directed towards a fixed point (i.e., mean position of rest) and is proportional to the displacement of the oscillating particle.**

Again, acceleration f is given by

$$f = \frac{d^2y}{dt^2} = -\omega^2 y$$

∴ acceleration = – (angular velocity)^2 × displacement

or, $$\text{angular velocity}^2 = \frac{\text{acceleration}}{\text{displacement}}$$ (considering the magnitude only)

or, $$\text{angular velocity} = \sqrt{\frac{\text{acceleration}}{\text{displacement}}}$$

It implies that,

$$\text{angular velocity} = \sqrt{\frac{\text{acceleration}}{\text{displacement}}}$$

Symbolically, we can write,

$$\omega = 2\pi\nu = \sqrt{\frac{f}{y}}$$ where ν is the frequency of oscillation.

or, $$\frac{2\pi}{T} = \sqrt{\frac{f}{y}},$$ where T is the time period of vibration

or, $$T = 2\pi\sqrt{\frac{y}{f}}$$

or, $$T = 2\pi\sqrt{k}$$

where k is the displacement per unit acceleration.

1.3 DIFFERENTIAL EQUATION OF SIMPLE HARMONIC MOTION

Let us consider an oscillating particle which is executing simple harmonic motion. The general equation of its displacement is given by,

$$y = a\sin(\omega t + \delta) \qquad \text{...(1.3)}$$

where y is the displacement and a is the amplitude and δ is the phase (or epoch) of the oscillating particle. Now, differentiating Eq. (1.3) with respect to time t, we get,

$$\frac{dy}{dt} = a\omega\cos(\omega t + \delta)$$

Here $\frac{dy}{dt}$ $(= v)$ represents the velocity of the particle.

Again, differentiating $\frac{dy}{dt}$ with respect to time t, we get

$$\frac{d^2y}{dt^2} = -a\omega^2\sin(\omega t + \delta)$$

But, we know that $y = a\sin(\omega t + \delta)$

$$\therefore \quad \frac{d^2y}{dt^2} = -\omega^2 y$$

or, $$\frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad \text{...(1.7)}$$

The second derivative $\frac{d^2y}{dt^2}$ represents the acceleration of the oscillating particle. Equation (1.7) represents the differential equation of simple harmonic motion.

In any phenomenon, where an equation similar to Eq. (1.7) is obtained, the related body executes simple harmonic motion (SHM). The general solution of Eq. (1.7) is given by

$$y = a\sin(\omega t + \delta)$$

One can also calculate the time period of oscillation of Eq. (1.7),

Numerically, $$\omega = \sqrt{\frac{d^2y/dt^2}{y}}$$

or, $$\omega = \sqrt{\frac{\text{acceleration}}{\text{displacement}}}$$

$$\therefore \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

1.4 VARIOUS CHARACTERISTICS OF SHM

When one particle executes simple harmonic motion, one can observe the following characteristics in its motion:

(a) The motion is periodic as well as oscillatory.

(b) The restoring force acting on the particle (and hence the acceleration) is directly proportional to the displacement of the particle and directs oppositely to the displacement measured from the mean position.

(c) The acceleration of the oscillating particle is always directed towards the mean position of its path.

(d) The motion of the simple harmonic oscillator (SHO) always takes place in a straight line.

1.5 SOLUTION OF THE DIFFERENTIAL EQUATION OF SHM

Let $y = ae^{\alpha t}$. Now differentiating y with respect to t, we get,

$$\frac{dy}{dt} = \alpha\, ae^{\alpha t}$$

or,
$$\frac{d^2y}{dt^2} = \alpha^2\,(ae^{\alpha t}) = \alpha^2 y$$

Now, substituting this value in Eq. (1.7), we get,

$$\alpha^2 y + \omega^2 y = 0$$

or,
$$\alpha^2 = -\,\omega^2$$

or,
$$\alpha = \pm\, i\omega \quad \text{where } i = \sqrt{-1}$$

So,
$$y = ae^{i\omega t} \quad \text{or} \quad y = ae^{-i\omega t}$$

Hence, the general solution is given by

$$y = a_1\, e^{+i\omega t} + a_2\, e^{-i\omega t} \qquad \text{...(1.8)}$$

where a_1 and a_2 are arbitrary constants.

$\therefore$
$$y = a_1\,(\cos \omega t + i \sin \omega t) + a_2\,(\cos \omega t - i \sin \omega t)$$

$$[\because e^{i\theta} = \cos\theta + i\sin\theta \text{ and } e^{-i\theta} = \cos\theta - i\sin\theta]$$

or,
$$y = (a_1 + a_2)\cos \omega t + i\,(a_1 - a_2)\sin \omega t$$

or,
$$y = a\cos \omega t + b\sin \omega t \qquad \text{...(1.9)}$$

where $a = (a_1 + a_2)$ and $b = i(a_1 - a_2)$

Let us now replace a and b by q and δ where $a = q\cos\delta$ and $b = q\sin\delta$

$\therefore$
$$q = \sqrt{a^2 + b^2}$$

and
$$\delta = \tan^{-1}\left(\frac{b}{a}\right)$$

Now, putting these values of a and b in Eq. (1.9), we get,

$$y = q\cos\delta\cos\omega t + q\sin\delta\sin\omega t$$

or,
$$y = q\cos(\omega t - \delta) \qquad \text{...(1.10)}$$

Hence, Eq. (1.10) represents the general solution of the differential equation

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

The maximum value of the displacement y is given by

$$y_{max} = q$$

q is known as amplitude of the simple harmonic motion.

If T be the period of oscillation, the same displacement repeats after an interval of time T, i.e.,

$$y_t = y_{t+T}$$

or, $$a\cos(\omega t - \delta) = a\cos\{\omega(t+T) - \delta\}$$

or, $$a\cos(\omega t - \delta) = a\cos\{(\omega t - \delta) + \omega T\}$$

or, $$\cos(\omega t - \delta) = \cos\{(\omega t - \delta) + \omega T\}$$

Hence, $$\omega T = 2\pi$$

$\therefore$ $$T = \frac{2\pi}{\omega} \quad ...(1.10)$$

or, $$\omega = \frac{2\pi}{T} \quad [\omega \text{ is angular velocity}]$$

The frequency of SHM is given by

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} \quad ...(1.11)$$

At $t = 0$, $y = a\cos\delta$

$\therefore$ $$\delta = \cos^{-1}\left(\frac{y}{a}\right)$$

The variable δ is known as the initial phase or epoch.

1.6 VELOCITY AND ACCELERATION OF THE PARTICLE EXECUTING SHO

The displacement (y) of a particle, which executes simple harmonic motion, at any time t is given by the following generalized equation,

$$y = a\sin(\omega t + \delta)$$

$\therefore$ the velocity of the particle, v is given by

$$v = \frac{dy}{dt} = a\omega\cos(\omega t + \delta) \quad ...(1.12)$$

When the particle reaches its mean position, the phase $\delta = 0$

$\therefore$ $$v = a\omega\cos(\omega t) = \pm a\omega\sqrt{1 - \sin^2 \omega t}$$

or, $$v = \pm a\omega\sqrt{1 - \frac{y^2}{a^2}}$$

$\therefore$ $$v = \pm \omega\sqrt{a^2 - y^2} \quad \text{where } \omega = 2\pi\nu$$

When the particle is at its mean position, we get $y = 0$, so in this position the particle velocity is maximum and it is given by

$$v_{max} = \pm a\omega$$

And at either of the extreme positions (i.e., $y = \pm a$), velocity becomes minimum and it is given by

$$v_{min} = 0$$

From Eq. (1.12), we get

$$v = a\omega\cos(\omega t + \delta)$$

∴ the acceleration of the particle is given by

$$f = \frac{dv}{dt} = -a\omega^2 \sin(\omega t + \delta) \qquad \text{...(1.13)}$$

When the particle is at its mean position, we get $\delta = 0$, so in this position the acceleration of the particle is given by

$$f = -a\omega^2 \sin \omega t$$

or,
$$f = -\omega^2 y \quad [\because t = 0 \text{ and } \delta = 0 \Rightarrow y = a \sin \omega t]$$

So, when the particle is at its extreme position, $y = a$ and the magnitude of the acceleration is given by

$$f_{max} = a\omega^2 \quad \text{[It is maximum]}$$

And when the particle is at its mean position, $y = 0$ and the magnitude of the acceleration is given by

$$f_{min} = 0 \quad \text{[It is minimum]}$$

1.7 ENERGY OF A PARTICLE EXECUTING SHM AND LAW OF CONSERVATION OF ENERGY

When a particle executes simple harmonic motion, it possesses both kinds of mechanical energy, namely potential and kinetic energy at any instant of time.

Kinetic Energy (E_k) In case of a particle of mass m and velocity v which is executing SHM, the kinetic energy is given by

$$E_k = \frac{1}{2} mv^2$$

But,
$$y = a \sin(\omega t + \delta)$$

∴
$$v = \frac{dy}{dt} = a\omega \cos(\omega t + \delta) = \pm a\omega \sqrt{1 - \frac{y^2}{a^2}}$$

or,
$$v = \pm \omega \sqrt{a^2 - y^2}$$

∴
$$E_k = \frac{1}{2} m\omega^2 (a^2 - y^2) \qquad \text{...(1.14)}$$

Potential Energy (E_P) In case of a particle of mass m and velocity v which is executing SHM, the potential energy is given by

$$E_p = \int_o^y (m\omega^2 y)\, dy = \frac{1}{2} m\omega^2 y^2 \qquad \text{...(1.15)}$$

Because the potential energy of an oscillating particle at any instant of time can be calculated from the total amount of work done in overcoming the effect of the restoring force, in this case the restoring force is given by,

$$F = -m\omega^2 y$$

When the particle comes to an extreme point, its kinetic and potential energies become

$$E_k = E_{k(\min)} = 0$$

and
$$E_p = E_{p(\max)} = \frac{1}{2} m\omega^2 a^2 \quad [\because y = a]$$

Similarly, when it is at the mean position, its kinetic and potential energies are given by

$$E_k = E_{k(\max)} = \frac{1}{2} m\omega^2 a^2$$

and
$$E_p = E_{p(\min)} = 0$$

So, the kinetic energy is maximum when the potential energy is minimum and vice versa.

Total Energy At any instant of time, the total energy of the oscillating particle is given by,

$$E = E_k + E_p$$

or,
$$E = \frac{1}{2} m\omega^2 (a^2 - y^2) + \frac{1}{2} m\omega^2 y^2$$

or, $E = \frac{1}{2} ma^2\omega^2 = 2\pi^2 ma^2 v^2$ ...(1.16)

$\therefore$ E = constant where $\omega = 2\pi v$

Hence, we can conclude that in case of simple harmonic oscillation the total energy is conserved. And it is proportional to the square of the amplitude of oscillation. The energy distribution of a simple harmonic oscillator has been shown in Fig. 1.2. Both kinetic and potential energies have been plotted against displacement.

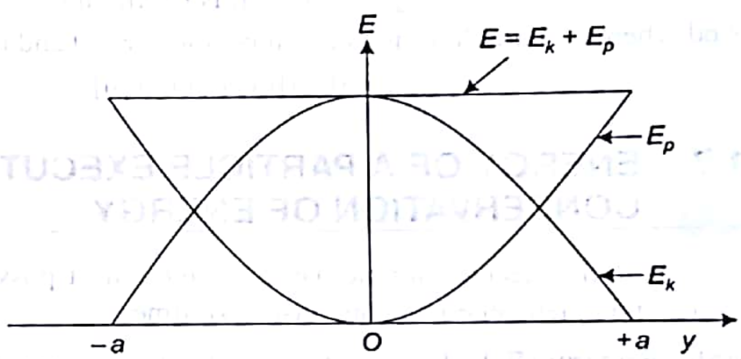


Fig. 1.2 **The energy distribution curve of an SHO. y is displacement, E_p is potential energy and E_k is the kinetic energy.**

1.8 SUPERPOSITION OF WAVES

It is an experimental fact that two or more waves can traverse through the same space independent of each other. For this reason, the displacement of any particle at a given instant of time is simply the sum of the individual displacements, this process of vector addition of the displacement of a particle is known as superposition.

Principle of Superposition **If the wave equations governing the wave motions are linear, the displacement of any particle, at a particular instant of time, is simply the algebraic sum of the individual displacements due to the different waves.**

1.8.1 Superposition of Two Collinear SHMs of Frequency v

Let two collinear simple harmonic motions be represented by the following two equations,

$$y_1 = a_1 \cos(\omega t - \phi_1)$$

and
$$y_2 = a_2 \cos(\omega t - \phi_2)$$

where y_1 and y_2 are instantaneous displacements, a_1 and a_2 are the amplitudes, the ω (= $2\pi v$) is, the angular velocity and ϕ_1 and ϕ_2 are the phases. The resultant amplitude at any instant of time is given by,

$$y = y_1 + y_2$$

or,
$$y = a_1 \cos(\omega t - \phi_1) + a_2 \cos(\omega t - \phi_2)$$

or,
$$y = a_1 \cos \omega t \cos \phi_1 + a_1 \sin \omega t \sin \phi_1 + a_2 \sin \omega t \cos \phi_2 + a_2 \sin \omega t \sin \phi_2$$

or,
$$y = (a_1 \cos \phi_1 + a_2 \cos \phi_2) \cos \omega t + (a_1 \sin \phi_1 + a_2 \sin \phi_2) \sin \omega t$$

Now, putting $A \cos \phi = a_1 \cos \phi_1 + a_2 \cos \phi_2$

and $$A \sin \phi = a_1 \sin \phi_1 + a_2 \sin \phi_2$$

where A and ϕ are two constants and given by,

$$A = \sqrt{(a_1 \cos \phi_1 + a_2 \cos \phi_2)^2 + (a_1 \sin \phi_1 + a_2 \sin \phi_2)^2}$$

or, $$A = \sqrt{a_1^{\,2} + a_2^{\,2} + 2a_1 a_2 \cos(\phi_1 - \phi_2)}$$

and $$\phi = \tan^{-1}\left(\frac{a_1 \sin\phi_1 + a_2 \sin \phi_2}{a_1 \cos \phi_1 + a_2 \cos \phi_2}\right)$$

$\therefore$ $$y = A \cos \phi \cos \omega t + A \sin \phi \sin \omega t$$

or, $$y = A \cos(\omega t - \phi) \qquad ...(1.17)$$

Thus, it is observed that the resultant motion is also a simple harmonic motion having the same component motions

Special Cases

(a) If $\phi_1 = \phi_2$ then $A = a_1 + a_2$

(b) If $\phi_1 \sim \phi_2 = \dfrac{\pi}{2}$ then $A = \sqrt{a_1^2 + a_2^2}$

(c) If $\phi_1 \sim \phi_2 = \pi$ then $A = a_1 \sim a_2$

(d) If $\phi_1 \sim \phi_2 = \pi$ and $a_1 = a_2$ then $A = 0$, i.e., the two SHMs destroy each other.

If instead of two SHMs, there remain several motions of different amplitudes and phases but of the same time period $T\left(=\dfrac{2\pi}{\omega}\right)$ then the resultant motion can be obtained in the same process. In such cases we will get

$$y = y_1 + y_2 + y_3 + ... + y_n$$

or, $$y = a_1 \cos(\omega t - \phi_1) + a_2 \cos(\omega t - \phi_2) + a_3 \cos(\omega t - \phi_3) + ... + a_n \cos(\omega t - \phi_n)$$

or, $$y = (a_1 \cos \phi_1 + a_2 \sin \phi_2 + a_3 \cos \phi_3 + ... + a_n \cos \phi_n) \cos \omega t + (a_1 \sin \phi_1 + a_2 \sin \phi_2 \sin \phi_3 + ... + a_n \sin \phi_n) \sin \omega t$$

or, $$y = A \cos(\omega t - \phi) \qquad ...(1.17)$$

where $$A = \sqrt{\left(\sum_{i=1}^{n} a_i \cos \phi_i\right)^2 + \left(\sum_{i=1}^{n} a_i \sin \phi_i\right)^2}$$

and $$\phi = \tan^{-1}\left(\frac{\sum_{i=1}^{n} a_i \sin \phi_i}{\sum_{i=1}^{n} a_i \cos \phi_i}\right)$$

1.8.2 Superposition of Two Mutually Perpendicular SHMs of Frequency ν

Let us consider two SHMs of amplitudes a_1 and a_2 having same angular velocity ω $(= 2\pi\nu)$ which act respectively along x- and y-axes.

Then their amplitudes can be represented by,

$$x = a_1 \cos(\omega t - \phi_1)$$

and $$y = a_2 \cos(\omega t - \phi_2)$$

Now, we can write,

$$\frac{y}{a^2} = \cos(\omega t - \phi_2)$$

$$= \cos\{\omega t - \phi_1) + (\phi_1 - \phi_2)\}$$

or, $$\frac{y}{a_2} = \cos(\omega t - \phi_1)\cos(\phi_1 - \phi_2) - \sin(\omega t - \phi_1)\sin(\phi_1 - \phi_2)$$

or, $$\frac{y}{a_2} = \frac{x}{a_1}\cos(\phi_1 - \phi_2) - \sqrt{\left(1 - \frac{x^2}{a_1^2}\right)}\sin(\phi_1 - \phi_2)$$

or, $$\frac{y}{a^2} - \frac{x}{a_1}\cdot\cos(\phi_1 - \phi_2) = -\sqrt{\left(1 - \frac{x^2}{a_1^2}\right)}\sin(\phi_1 - \phi_2)$$

Squaring both sides of the equation, we get

$$\frac{y^2}{a_2^2} + \frac{x^2}{a_1^2}\cos^2(\phi_1 - \phi_2) - \frac{2xy}{a_1 a_2}\cos(\phi_1 - \phi_2) = \left(1 - \frac{x^2}{a_1^2}\right)\sin^2(\phi_1 - \phi_2)$$

or, $$\frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2}\cos(\phi_1 - \phi_2) + \frac{x^2}{a_1^2}\{\cos^2(\phi_1 - \phi_2) + \sin^2(\phi_1 - \phi_2)\} = \sin^2(\phi_1 - \phi_2)$$

or, $$\frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2}\cos(\phi_1 - \phi_2) + \frac{x^2}{a_1^2} = \sin^2(\phi_1 - \phi_2)$$

or, $$\frac{y^2}{a_2^2} - 2\frac{xy}{a_1 a_2}\cos\phi + \frac{x^2}{a_1^2} = \sin^2\phi \qquad \text{...(1.18)}$$

where $\phi = \phi_1 - \phi_2$

Equation (1.18) is a general equation of an ellipse bounded within a rectangle with sides $2a_1$ and $2a_2$. Thus the resultant motion (having combined two mutually perpendicular SHMs) is represented by an ellipse. Let us now consider the special cases of this generalized motion.

Case 1 If $\phi_1 - \phi_2 = 0$ or $\phi = 0$ then $\sin\phi = 0$ and $\cos\phi = 1$

$\therefore$ Eq. (1.18) gets reduced to

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - 2\frac{xy}{a_1 a_2} = 0$$

or, $$\left(\frac{x}{a_1} - \frac{y}{a_2}\right)^2 = 0$$

or, $$y = \frac{a_2}{a_1}x$$

This represents a straight line passing through the origin and making an angle of inclination $\delta = \tan^{-1}\frac{a_2}{a_1}$ to the x-axis (Fig. 1.3).

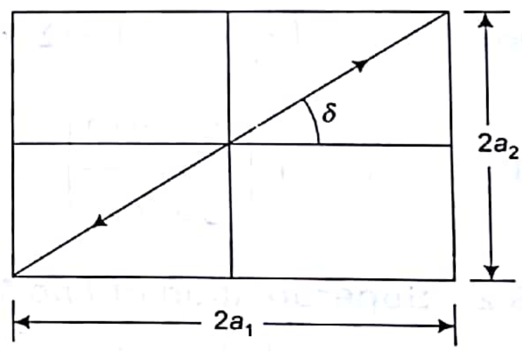


Fig. 1.3 **The resultant vibration of two mutually perpendicular vibrations. It takes place in a straight line which is inclined at an angle δ with the x-axis and $\delta < \frac{\pi}{2}$ when $\phi = 0$.**

Case 2 If $\phi_1 - \phi_2 = \pi/2$ or $\phi = \frac{\pi}{2}$, then $\sin\phi = 1$ and $\cos\phi = 0$.

Equation (1.18) reduces to

$$\frac{x^2}{a_1^2}+\frac{y^2}{a_2^2}=1$$

This equation represents an ellipse which is symmetrical about the two axes and a_1 and a_2 are the semi-axes (Fig. 1.4).

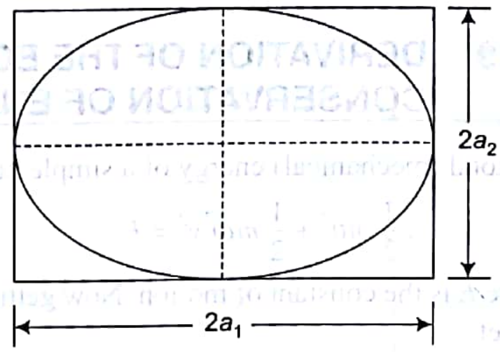


Fig. 1.4 The resultant of two mutually perpendicular vibrations represents an ellipse.

Case 3 If $\phi_1 - \phi_2 = \pi$ or $\phi = \pi$ then $\sin\phi = 0$ and $\cos\phi = -1$

$\therefore$ Eq. (1.18) gets reduced to

$$\frac{y^2}{a_2^2}+2\frac{xy}{a_1 a_2}+\frac{x^2}{a_1^2}=0$$

or,
$$\left(\frac{x}{a_1}+\frac{y}{a_2}\right)^2=0$$

or,
$$y=-\frac{a_2}{a_1}x$$

This equation represents a straight line which passes through the origin making an angle of inclination $\delta=\tan^{-1}\left(-\frac{a_2}{a_1}\right)$ to the x-axis (Fig. 1.5).

Case 4 If $\phi_1 - \phi_2 = \pi/2$ or $\phi = \pi/2$ and $a_1 = a_2$ $(=a)$ (say) then $\sin\phi = 1$ and $\cos\phi = 0$.

In this case, Eq. (1.18) gets reduced to

$$x^2+y^2=a^2$$

This equation represents a circle which is symetrical about the two axes with a as radius (Fig. 1.6).

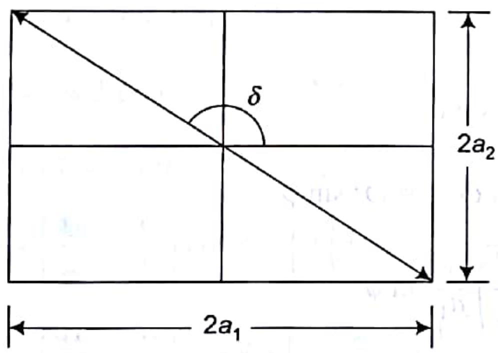


Fig. 1.5 The resultant of two mutually perpendicular vibrations represents a straight line which is inclined at an angle δ where $\delta > \frac{\pi}{2}$ when $\phi = \pi$.

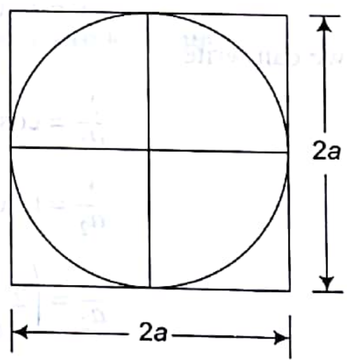


Fig. 1.6 The resultant of two mutually perpendicular vibrations represents a circle when $\phi = \frac{\pi}{2}$ and $a_1 = a_2$.

1.9 DERIVATION OF THE EQUATION OF MOTION FROM THE LAW OF CONSERVATION OF ENERGY

The total (mechanical) energy of a simple harmonic oscillator is given by,

$$\frac{1}{2}mv^2 + \frac{1}{2}m\omega^2y^2 = E \qquad ...(1.19)$$

where E is the constant of motion. Now getting differentiated on both sides of Eq. (1.19) with respect to time, we get,

$$mv\frac{dv}{dt} + m\omega^2 y\frac{dy}{dt} = 0$$

or,

$$mv\left(\frac{dv}{dt} + \omega^2 y\right) = 0$$

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) + \omega^2 y = 0$$

or,

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad ...(1.20)$$

Equation (1.20) is the same differential equation as Eq. (1.7).

1.10 TWO VIBRATIONS IN A PLANE (OF COMMENSURATE FREQUENCIES) ACTING AT RIGHT ANGLES TO EACH OTHER

Case 1 If the frequencies of the two vibrations be of the ratio 1:2 with initial phase difference then we can represent the two vibrations by the following two equations:

$$x = a_1 \cos(\omega t)$$

and

$$y = a_2 \cos(2\omega t + \phi)$$

Now, we can write,

$$\frac{y}{a_2} = \cos(2\,\omega t)\cos\phi - \sin(2\,\omega t)\sin\phi$$

or,

$$\frac{y}{a_2} = (2\cos 2\,\omega t - 1)\cos\phi - 2\sin\omega t\cos\omega t\sin\phi$$

or,

$$\frac{y}{a_2} = \left(2\frac{x^2}{a_1^2} - 1\right)\cos\phi - 2\sqrt{\left(1 - \frac{x^2}{a_1^2}\right)}\frac{x}{a_1}\sin\phi$$

or,

$$\frac{y}{a_2} - \left(\frac{2x^2}{a_1^2} - 1\right)\cos\phi = -2\sqrt{\left(1 - \frac{x^2}{a_1^2}\right)}\frac{x}{a_1}\sin\phi$$

Now, having squared both sides of the equation, we get,

$$\frac{y^2}{a_2^2} - 2\frac{y}{a_2}\left(\frac{2x^2}{a_1^2} - 1\right)\cos\phi + \left(\frac{2x^2}{a_1^2} - 1\right)^2\cos^2\phi = 4\left(1 - \frac{x^2}{a_1^2}\right)\frac{x^2}{a_1^2}\sin^2\phi$$

or, $$\frac{4x^2}{a_1^2}\left(\frac{x^2}{a_1^2}-\frac{y}{a_2}\cos\phi-1\right)+\left(\frac{y}{a_2}+\cos\phi\right)^2=0 \qquad ...(1.21)$$

The above equation gives the general equation of the resultant motion for any phase difference and amplitudes. If $\phi = 0$, Eq. (1.21) gets reduced to

$$\left(\frac{2x^2}{a_1^2}-\frac{y}{a_2}-1\right)^2=0 \qquad ...(1.22)$$

Equation (1.22) represents two coincident parabolas (Fig. 1.7).

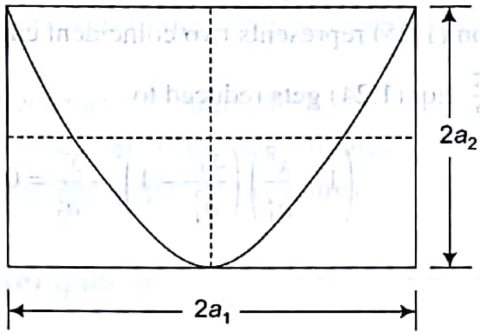


Fig. 1.7 **Resultant of two mutually perpendicular vibrations with frequency ratio = 1 : 2 and phase difference $\phi = 0$ represents coincident parabolas.**

If $\phi = \frac{\pi}{2}$, Eq. (1.21) gets reduced to

$$\frac{4x^2}{a_1^2}\left(\frac{x^2}{a_1^2}-1\right)+\frac{y^2}{a_2^2}=0 \qquad ...(1.23)$$

Equation (1.23) is a 4th degree equation and represents a curve which has two loops as has been shown in Fig. 1.8.

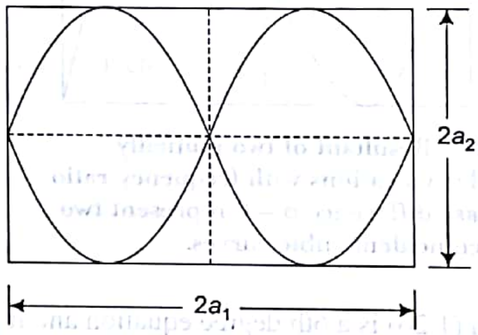


Fig. 1.8 **Resultant of two mutually perpendicular vibrations with frequency ratio 1 : 2 and phase difference $\phi = \frac{\pi}{2}$ represents a two loop curve.**

Case 2 If the frequencies of two vibrations be of the ratio 1 : 3 with initial phase difference ϕ then we can represent the two vibrations by the following two equations:

$$x = a_1 \cos(\omega t)$$

and $$y = a_2 \cos(3\omega t + \phi)$$

Now, we can write,

$$\frac{y}{a_2} = \cos(3\omega t + \phi)$$

or, $$\frac{y}{a_2} = \cos 3\,\omega t \cos\phi - \sin 3\omega t \sin\phi$$

or, $$\frac{y}{a_2} = (4\cos^3\omega t - 3\cos\omega t)\cos\phi - (3\sin\omega t - 4\sin^3\omega t)\sin\phi$$

or, $$\frac{y}{a_2} = \left(\frac{4x^3}{a_1^3}-\frac{3x}{a_1}\right)\cos\phi - \left\{3\left(1-\frac{x^2}{a_1^2}\right)^{\frac{1}{2}} - 4\left(1-\frac{x^2}{a_1^2}\right)^{\frac{3}{2}}\right\}\sin\phi$$

or, $$\frac{y}{a_2} = \left(\frac{4x^3}{a_1^3}-\frac{3x}{a_1}\right)\cos\phi - \left(1-\frac{x^2}{a_1^2}\right)^{\frac{1}{2}}\left(\frac{4x^2}{a_1^2}-1\right)\sin\phi \qquad ...(1.24)$$

Now, transposing and squaring, we get,

$$\left\{\frac{y}{a_2}-\left(\frac{4x^3}{a_1^3}-\frac{3x}{a_1}\right)\cos\phi\right\}^2 = \left(1-\frac{x^2}{a_1^2}\right)\left(\frac{4x^2}{a_1^2}-1\right)^2\sin^2\phi$$

If $\phi = 0$, then Eq. (1.24) gets reduced to

$$\left\{\frac{y}{a_2}-\left(\frac{4x^3}{a_1^3}-\frac{3x}{a_1}\right)\right\}^2=0 \qquad \text{...(1.25)}$$

Equation (1.25) represents two coincident cubic curves (Fig. 1.9).

If $\phi=\frac{\pi}{2}$, Eq. (1.24) gets reduced to

$$\left(1-\frac{x^2}{a_1^2}\right)\left(\frac{4x^2}{a_1^2}-1\right)^2-\frac{y^2}{a_2^2}=0 \qquad \text{...(1.26)}$$

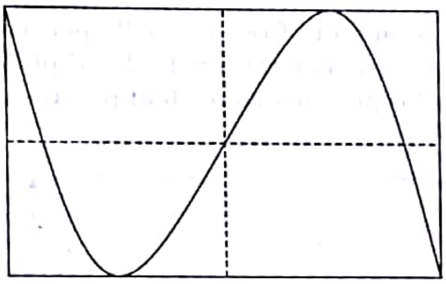


Fig. 1.9 Resultant of two mutually perpendicular vibrations with frequency ratio 1:3 and phase difference $\phi = 0$ represent two coincident cubic curves.

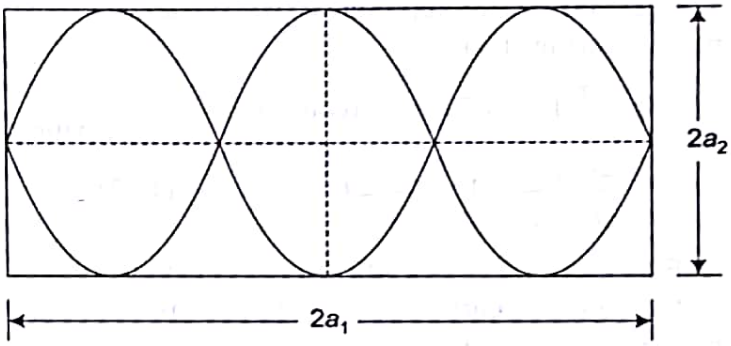


Fig. 1.10 Resultant of two mutually perpendicular vibrations with frequency ratio 1:3 and phase difference $\phi=\frac{\pi}{2}$ represents a curve of three loops.

Equation (1.26) is a 6th degree equation and it gives a curve of three loops (Fig. 1.10).

If one changes the phase difference gradually, then the shape of the loop gradually changes.

For a ratio of frequencies of $1:n$, the curve will have n loops.

1.10.1 Lissajous Figures

Definition Lissajous figures are those curves which are generated by superimposing two simple harmonic motion acting at right angles to each other.

The constituent simple harmonic motions may have different time periods, different amplitudes and also different initial phases. The size (or dimension) of the resultant curve depends on their amplitudes but the shape of the curve depends on the ratio of their time periods and the initial phase differences. These figures may be experimentally generated by (a) Blackburn's pendulum, (b) optical method, and (c) cathode ray oscillograph, etc. The diagram of an oscillograph is shown in Fig. 1.11.

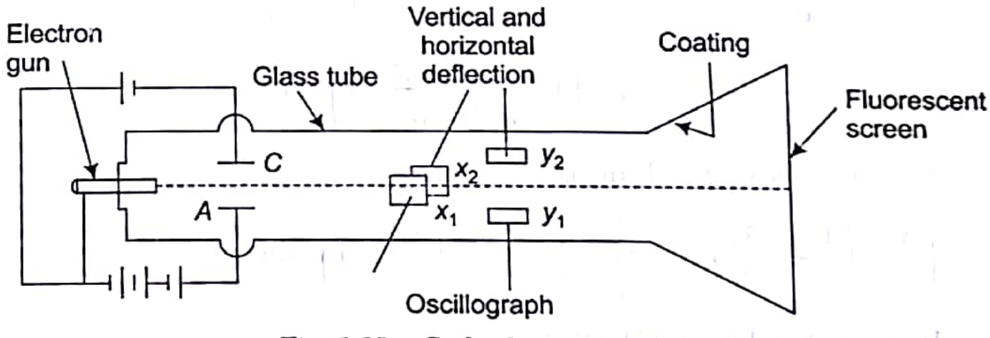


Fig. 1.11 Cathode-ray oscillograph.

Figures 1.7, 1.8, 1.9 and 1.10 are examples of Lissaious figures.

Uses of Lissajous Figures

(a) Lissajous figures can be used for determination of the ratio of the frequencies of two mutually perpendicular superposing vibrations.

(b) They also can be used to determine the unknown frequency of a tuning fork.

(c) The nature and shape of a signal can be known with the help of Lissajous' figures.

(d) The amplitude of a signal also can be determined with the help of these figures.

Worked-out Examples

Example 1.1 The displacement of a body of mass 2 g executing simple harmonic motion is indicated by $y = 10 \sin\left(\frac{\pi}{3}t + \frac{\pi}{15}\right)$ cm. Calculate the (a) amplitude, (b) angular velocity, (c) time period, (d) maximum and minimum velocity and maximum and minimum acceleration, (e) epoch, (f) kinetic energy, and (g) potential energy. Is its energy conserved?

Sol. The general equation of a simple harmonic oscillator is given

$$y = a \sin(\omega t - \delta) \quad \text{...(1)}$$

And in the present case, it is

$$y = 10 \sin\left(\frac{\pi}{3}t + \frac{\pi}{15}\right) \quad \text{...(2)}$$

Now, comparing eqs. (1) and (2) we get,

(a) The amplitude, $a = 10$ cm,

(b) The angular velocity, $\omega = \frac{\pi}{3}$ rad s^{-1}

(c) The time period, $T = \frac{2\pi}{\omega}$ or, $T = 2\pi \times \frac{3}{\pi} = 6$ s

(d) The velocity of the oscillating body at any time is given by $v = \pm\omega\sqrt{a^2 - y^2}$

Hence,
$$v_{max} = \left|\pm \omega \sqrt{a^2 - y^2}\right|_{max}$$
$$= |\pm \omega a| \quad [\text{for } y = 0]$$

i.e.,
$$v_{max} = a\omega = 10 \times \frac{\pi}{3} = 10.46 \text{ cm/s}$$

Similarly, the minimum velocity is given by $v_{min} = 0$ cm s^{-1} [for $y = a$]

The acceleration at any time t is given by

$$f = \omega^2 y$$

$\therefore$
$$f_{max} = a\omega^2 \quad [\text{for } y = a]$$

or,
$$f_{max} = 10 \times \frac{\pi^2}{9} = 10 \times \left(\frac{3.14}{3}\right)^2$$

or,
$$f_{max} = 10.95 \text{ cm s}^{-2}$$

And the minimum acceleration f_{min} is given by

$$f_{min} = 0 \text{ cm s}^{-1} \quad [\text{for } y = 0]$$

(e) The phase of the oscillating body is given by,

$$\phi = (\omega t - \delta)$$

Now, we know that epoch is the initial phase at $t = 0$

$\therefore$ from Eq. (2), we get $\delta = -\dfrac{\pi}{15}$

(f) The kinetic energy is given by

$$E_k = \frac{1}{2} m\omega^2 (a^2 - y^2)$$

So, the value of kinetic energy will vary with y.

(g) The potential energy is given by

$$E_p = \frac{1}{2} m\omega^2 y^2$$

$\therefore$ the value of the potential energy varies as y^2.

At any time, the total energy is given by

$$E = E_p + E_k$$

or,
$$E = \frac{1}{2} m\omega^2 y^2 + \frac{1}{2} m\omega^2 (a^2 - y^2)$$

or,
$$E = \frac{1}{2} m\omega^2 y^2 + \frac{1}{2} m\omega^2 a^2 - \frac{1}{2} m\omega^2 y^2$$

$\therefore$
$$E = \frac{1}{2} m\omega^2 a^2$$

which is independent of the variables t and y.

So, it is a constant w.r.t. time. Hence, the total energy of the oscillator is conserved.

Example 1.2 A particle is executing SHM. At an instant of time its displacement is 12 cm, velocity is 5 cm s^{-1} and when its displacement is 5 cm, the velocity is 12 cm s^{-1}. Calculate its (i) amplitude, (ii) frequency, and (iii) time period.

Sol. The velocity of a particle executing SHM is given by

$$v = \frac{dy}{dt} = \omega\sqrt{a^2 - y^2}$$

In the first case,

$$v_1 = \omega\sqrt{a^2 - y_1^2}$$

where $v_1 = 5$ cm s^{-1} and $y_1 = 12$ cm

or,
$$5 = \omega\sqrt{a^2 - 144} \quad \text{...(1)}$$

In the second case

$$v_2 = \omega\sqrt{a^2 - y_2^2}$$

where $v_2 = 12$ cm s^{-1}, $y_2 = 5$ cm

$\therefore$
$$12 = \omega\sqrt{a^2 - 25} \quad \text{...(2)}$$

Dividing (1) by (2), we get

$$\frac{5}{12} = \frac{\sqrt{a^2 - 144}}{\sqrt{a^2 - 25}}$$

or,
$$\frac{25}{144} = \frac{a^2 - 144}{a^2 - 25}$$

$\therefore$ the amplitude is 13 cm.

Now, substituting the value of $a = 13$ cm in Eq. (1), we get,

$$5 = \omega\sqrt{13^2 - 144}$$

or,
$$5 = \omega\sqrt{25} \quad \therefore\ \omega = 1 \text{ rad s}^{-1}$$

The frequency $\nu = \dfrac{\omega}{2\pi} = \dfrac{1}{2\pi}$ Hz

The time period is $T = \dfrac{1}{\nu} = 2\pi$ s

Example 1.3 Show that for a particle executing SHM, its velocity at any instant of time is given by

$$\frac{dy}{dt} = \omega\sqrt{a^2 - y^2}$$

Sol. The displacement of a particle executing SHM is given by,

$$y = a \sin(\omega t + \phi) \qquad ...(1)$$

The velocity at any instant of time is given by

$$\frac{dy}{dt} = a\omega \cos(\omega t + \phi) \qquad ...(2)$$

or,
$$\frac{dy}{dt} = a\omega\sqrt{1 - \sin^2(\omega t + \phi)}$$

or,
$$\frac{dy}{dt} = \omega\sqrt{a^2 - \{a \sin(\omega t + \phi)\}^2}$$

or,
$$\frac{dy}{dt} = \omega\sqrt{a^2 - y^2} \quad \text{[by Eq. (1)]}$$

Example 1.4 The motion of a particle in SHM is given by $y = a \sin \omega t$. If it has a speed u when the displacement is y_1, and a speed v when the displacement is y_2, show that the amplitude of the motion is

$$a = \left[\frac{v^2y_1^2 - u^2y_2^2}{v^2 - u^2}\right]^{\frac{1}{2}}$$

Sol. We have, $y = a \sin \omega t$

$\therefore$
$$u = \frac{dy_1}{dt} = \omega\sqrt{a^2 - y_1^2} \qquad ...(1)$$

and
$$v = \frac{dy_2}{dt} = \omega\sqrt{a^2 - y_2^2} \qquad ...(2)$$

Now, squaring both the equations, and dividing we have

$$\frac{u^2}{v^2} = \frac{a^2 - y_1^2}{a^2 - y_2^2}$$

or, $$u^2a^2 - u^2y_2^2 = v^2a^2 - v^2y_1{}^2$$

or, $$a^2\,[u^2 - v^2] = u^2y_2^2 - v^2y_1^2$$

or, $$a^2 = \frac{u^2\,y_2^2 - v^2y^2{}_1}{u^2 - v^2}$$

or, $$a = \left[\frac{v^2y^2{}_1 - u^2y_2^2}{v_2 - u^2}\right]^{\frac{1}{2}}$$

Example 1.5 Show that for a particle executing SHM, the instantaneous velocity is $\omega\sqrt{a^2 - y^2}$ and the instantaneous acceleration is $-\,\omega^2 y$ where ω is the angular frequency, a is the amplitude and y is the instantaneous displacement.

Sol. The displacement of a particle executing SHM is given by,

$$y = a\cos(\omega t + \delta) \qquad \text{...(1)}$$

The instantaneous velocity is

$$v = \frac{dy}{dt} = -\,\omega a\sin(\omega t + \delta) \qquad \text{...(2)}$$

From Eq. (1)

$$\cos(\omega t + \delta) = \frac{y}{a}$$

$\therefore$ $$\sin(\omega t + \delta) = \sqrt{1 - \cos^2(\omega t + \delta)} = \sqrt{1 - \frac{y^2}{a^2}}$$

or, $$-\,a\omega\sin(\omega t + \delta) = -\,a\omega\sqrt{1 - \frac{y^2}{a^2}}$$

or, $$v = -\frac{a\omega}{\sqrt{a^2}}\left(\sqrt{a^2 - y^2}\right) \quad \text{[by Eq.(2)]}$$

$\therefore$ $$v = \pm\,\omega\sqrt{a^2 - y^2)}$$

Now, differentiating Eq. (2), we get

$$f = \frac{dv}{dt} = -\,\omega^2 a\cos(\omega t + \delta)$$

or, $$f = -\,\omega^2 y \quad \text{[by Eq. (1)]}$$

Example 1.6 Calculate the time period of the liquid column of length l in a U-tube, if it is depressed in one arm by x, d is the density of the liquid and A is the cross-sectional area of the arm of the U-tube.

[WBUT 2007]

Sol. The U-tube as has been described in the question has been shown in Fig. 1.12.

The liquid column of a liquid with density d has been shown in the diagram. The height of the liquid column in each arm is equal to l. The liquid column has been depressed in the left arm through a depth of x. So, in the right arm it has risen through a height of x. So the difference in the levels of the two arms is $2x$. The liquid column of length $2x$ in the right arm will try to bring back the two levels to their initial values. The force acting in this process will be

$$F = (2x)\,Adg$$

This force (F) acts on the liquid of both columns with length l.

$$\therefore \quad (2l\,Ad)\frac{d^2x}{dt^2} = -(2x)\,Adg \quad [\text{here, } m = 2l\,Ad]$$

or,
$$\frac{d^2x}{dt^2} = -\frac{xg}{l}$$

or,
$$\ddot{x} + \left(\frac{g}{l}\right)x = 0$$

$$\therefore \quad \ddot{x} + \omega^2 x = 0 \quad \left[\text{where } \omega^2 = \frac{g}{l}\right]$$

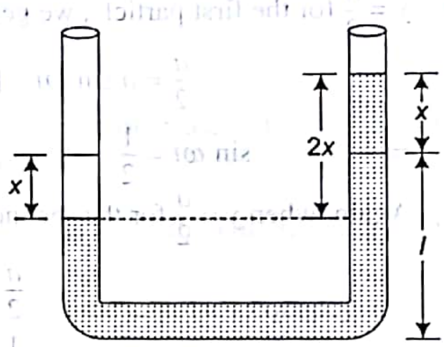


Fig. 1.12 A U-tube with vibrating liquid column.

This is the equation of a simple harmonic oscillator.

$\therefore$ its time-period T is given

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

Example 1.7 9 kg of mercury is poured into a glass U-tube with a uniform internal diameter of 1.2 cm. It oscillates freely about its equilibrium position. Calculate the time period of oscillation of the mercury column.

Sol. Let l be the height of the mercury column in each arm and m be the mass of liquid of the two columns. Then

$$m = 2l\,Ad \quad \text{where } A \text{ is the area of cross-section and } d \text{ is the density}$$

$$\therefore \quad l = \frac{m}{2Ad}$$

The mass $m = 9$ kg, $d = 13.6 \times 10^3$ kg m^{-3}

$$A = 3.14 \times \left(\frac{1.2}{2}\right)^2 \text{cm}^2 = 1.1304 \times 10^{-4}\ \text{m}^2$$

$$\therefore \quad l = \frac{9}{2 \times (1.1304 \times 10^{-4}) \times (13.6 \times 10^3)} = 2.927\ \text{m}$$

$\therefore$ the time period of oscillation T is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

or,
$$T = 2\pi \times \sqrt{\frac{2.927}{9.8}}$$

$$\therefore \quad T = 3.43\ \text{s}$$

Example 1.8 Two particles execute SHM of the same amplitude and frequency along two parallel straight lines. They pass one another when going in opposite directions. Each time their displacement is half of their amplitude. What is the phase difference between them?

Sol. Let the equation of motion of the two particles be as follows:

$$y = a \sin \omega t \quad \text{...(1)}$$

and

$$y = a \sin (\omega t + \phi) \quad \text{...(2)}$$

when $y = \frac{a}{2}$ for the first particle, we get

$$\frac{a}{2} = a \sin \omega t \quad \text{[from Eq. (1)]}$$

$$\Rightarrow \quad \sin \omega t = \frac{1}{2} \quad \text{...(3)}$$

Again, when $y = \frac{a}{2}$ for the second particle, we get, [from Eq. (2)]

$$\frac{a}{2} = a \sin (\omega t + \phi)$$

or, $$\frac{1}{2} = \sin \omega t \cos \phi + \cos \omega t \sin \phi$$

or, $$\frac{1}{2} = \frac{1}{2} \cos \phi + \sqrt{1 - \frac{1}{4}} \sin \phi \quad \text{[by Eq. (3)]}$$

or, $$1 = \cos \phi + \sqrt{3} \sin \phi$$

or, $$1 - \cos \phi = \sqrt{3} \sin \phi$$

or, $$1 - 2 \cos \phi + \cos^2 \phi = 3 \sin^2 \phi$$

or, $$1 - 2 \cos \phi + \cos^2 \phi = 3 - 3 \cos^2 \phi$$

or, $$4 \cos^2 \phi - 2 \cos \phi - 2 = 0$$

or, $$2 \cos^2 \phi - \cos \phi - 1 = 0$$

or, $$(2 \cos \phi + 1)(\cos \phi - 1) = 0$$

Now, $(\cos \phi - 1) = 0$ will give the value of ϕ which equals to zero. So, this value of ϕ cannot be accepted as the particles will have the same phase.

Hence, $$(2 \cos \phi + 1) = 0$$

or, $$\cos \phi = -\frac{1}{2} = \cos 120^\circ$$

Therefore, the phase difference between the two oscillators is 120°.

Example 1.9 The displacement of a moving particle at any time t is given by

$$y = a \sin \omega t + b \cos \omega t$$

Show that the motion is simple harmonic.

Sol. The equation representing the displacement of the particle is given by

$$y = a \sin \omega t + b \cos \omega t \quad \text{...(1)}$$

or, $$\frac{dy}{dt} = a\omega \cos \omega t - b\omega \sin \omega t$$

or, $$\frac{d^2y}{dt^2} = -a\omega^2 \sin \omega t - b\omega^2 \cos \omega t$$

or, $$\frac{d^2y}{dt^2} = -\omega^2 (a \sin \omega t + b \cos \omega t)$$

$$\frac{d^2y}{dt^2} = -\omega^2 y \quad \text{[by Eq. (1)]}$$

or, $$\frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad ...(2)$$

Equation (2) is the standard differential equation of a simple harmonic oscillator. Hence the given equation [i.e., Eq. (1)] represents a simple harmonic motion.

Example 1.10 Calculate the displacement to amplitude ratio for SHM when the kinetic energy is 90% of the total energy.

Sol. Let m be the mass of the oscillator, a be the amplitude and ω be the angular frequency.

So, the total energy of the oscillator is given by

$$E = \frac{1}{2} m\omega^2 a^2$$

Let y_1 be the displacement when its kinetic energy E_k is 90% of the total energy. Now, the potential energy is given by

$$E_p = \frac{1}{2} m\omega^2 y_1^2$$

$$\therefore \quad \frac{E_p}{E} = \frac{\frac{1}{2} m\omega^2 y_1^2}{\frac{1}{2} m\omega^2 a^2} = \frac{y_1^2}{a^2} = \frac{10}{100} = 0.1$$

$$\therefore \quad y_1 : a = \sqrt{0.1} = 0.316$$

Review Exercises

Part 1: Multiple Choice Questions

1. The SI unit of the force constant of a spring is given by
   (a) Nm (b) Nm^{-2} (c) Nm^{-1} (d) N
2. Which of the following is not essential for simple harmonic motion?
   (a) Inertia (b) Gravity (c) Restoring force (d) Elasticity
3. The velocity of a particle executing simple harmonic motion is minimum at a point where displacement is
   (a) zero (b) maximum
   (c) midway between zero and maximum (d) none of these

4. If two SHMs of the same amplitude, time period and phase act at right angles to each other, then the resultant vibration is

   (a) elliptical (b) circular (c) straight line (d) parabolic

5. If the velocity of a particle executing SHM is maximum, then displacement will be

   (a) maximum (b) minimum (c) less than zero (d) greater than zero

6. The potential energy of a particle executing SHM of amplitude a is equal to its kinetic energy when displacement of the particle is

   (a) $\pm a$ (b) $\pm \frac{a}{\sqrt{2}}$ (c) $\pm \frac{a}{2}$ (d) $\pm \frac{a}{4}$

7. For a particle executing SHM, the phase difference between displacement and velocity is

   (a) π (b) 0 (c) $\frac{\pi}{2}$ (d) $-\frac{\pi}{2}$

8. Which one of the following statements is true?

   (a) All periodic motion is simple harmonic.
   (b) All simple harmonic motions are periodic.
   (c) Potential energy is proportional to displacement.
   (d) All of the above statements are incorrect.

9. Which one of the following relations is not true for a SHM?

   (a) Potential energy is always equal to the kinetic energy
   (b) $T = 2\pi\sqrt{\frac{\text{Displacment}}{\text{Acceleration}}}$
   (c) $\frac{d^2y}{dt^2} + \frac{\beta}{m} y = 0$
   (d) none of these

10. The equation $\frac{d^2y}{dt^2} = -\omega^2 y$ represents (where ω is a constant)

    (a) projectile motion (b) motion of a freely falling body
    (c) simple harmonic motion (d) none of these

11. Production of Lissajous figures can be shown through a device known as

    (a) Geiger–Muller counter (b) travelling microscope
    (c) cathode ray oscilloscope (d) periscope

12. The differential equation of a simple harmonic motion of a particle of mass m with angular frequency ω can be expressed as

    (a) $\frac{d^2y}{dt^2} - \omega^2 y = 0$ (b) $\frac{d^2y}{dt^2} + \omega^2 y = 0$ (c) $\frac{md^2y}{dt^2} = 0$ (d) $\frac{md^2y}{dt^2} + \omega^2 y = 0$

13. The total mechanical energy of a particle (executing SHM) of mass m, angular frequency ω and amplitude of vibration is given by

    (a) $\frac{1}{2} m\omega^2$ (b) $\frac{p^2}{2m}$ (c) $\frac{1}{2} m\omega^2 a^2$ (d) $\frac{1}{2} ma^2$

14. To have a circular Lissajous figure, the phase difference (δ) and the amplitudes (a and b) of two superposing simple harmonic motions are respectively
    (a) $\delta = 0, a \neq b$ (b) $\delta = 0, a = b$ (c) $\delta = \pi/2, a = b$ (d) $\delta = \pi/2, a \neq b$
15. The time period of a particle of mass m (executing SHM) under, a force where the force per unit displacement is k can be expressed as
    (a) $T = 2\pi\sqrt{\frac{k}{m}}$ (b) $T = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ (c) $T = 2\pi\sqrt{\frac{m}{k}}$ (d) none of these
16. In any damped oscillation the resisting force is proportional to
    (a) displacement (b) acceleration (c) velocity (d) square of velocity
17. The general solution of the differential equation of simple harmonic motion is given by
    (a) $y = a \sin \omega t$ (b) $y = \cos \omega t$ (c) $y = a \cos \delta$ (d) $y = a \sin (\omega t + \delta)$
18. The sound wave is
    (a) an electromagnetic wave (b) an elastic wave
    (c) a radio wave (d) none of these
19. The time period of a simple pendulum of infinite length is given by
    (a) finite (b) zero (c) infinite (d) none of these
20. If the differential equation of the SHM of a body is represented by $\frac{d^2y}{dt^2} + \omega^2 y = 0$ then its natural frequency is given by
    (a) ω (b) $\frac{2\pi}{\omega}$ (c) $\frac{\omega}{2\pi}$ (d) $\frac{\omega}{\pi}$
21. If the restoring force constant of a body is 98 Nm^{-1}, then the restoring force of the body for a displacement of 10 cm is
    (a) 98 N (b) 9.8 N (c) 0.98 N (d) none of these

[**Ans.** 1. (c), 2. (b), 3. (c), 4. (b), 5. (b), 6. (b), 7. (c), 8. (b), 9. (a), 10.(c), 11. (c), 12. (b), 13. (c), 14. (c), 15. (c), 16. (c), 17. (d), 18. (b), 19. (c), 20. (c), 21 (b)]

Short Questions with Answers

1. Give four examples of non-oscillatory periodic motion.

Ans. The following are the examples of non-oscillatory periodic motion:

(a) The motion of the earth around the sun, (b) the motion the moon around the earth, (c) the motion of two twin stars, and (d) the whirling of a stone tied to a string.

2. On what factors does the shape of Lissajous' figures depend?

Ans. The shape of Lissajous' figures depends on the following factors:

(a) The ratio of the frequencies (or periods)
(b) The amplitudes of the oscillations
(c) The relative phase of the component motions

3. Why is a loaded bus more comfortable than an empty bus?

Ans. For a spring-mass system, the frequency of the vibrating mass is given by

$$\nu = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$$

where ν is the frequency of vibration, k is the spring constant and m is the mass. One can see from the equation that the frequency of vibration is inversely proportional to the square root of the mass. Hence, if the bus is loaded there will be less vibrations and it will be more comfortable to its riders.

4. Is a transformer a source of simple harmonic motion?

Ans. A transformer simply lowers the voltage of the applied ac supply and the secondary also generates an alternating current which reverses its direction according to simple harmonic motion. So, we can consider a transformer as a source of simple harmonic motion.

5. The potential energy of a particle of mass m is given by $\frac{1}{2}m\omega^2y^2$, where ω is a constant. Show that the particle is executing simple harmonic motion.

Ans. In the given problem, the potential energy is given by

$$V = \frac{1}{2}m\omega^2y^2$$

If F be the force acting on the particle then

$$F = -\frac{dV}{dy}$$

or,
$$F = -\frac{d}{dy}\left(\frac{1}{2}m\omega^2y^2\right) = -m\omega^2y$$

Hence, the acceleration of the particle is given by

$$f = \frac{F}{m} = -\omega^2y$$

or,
$$f \propto -y \quad \text{(as } \omega \text{ is a constant)}$$

Hence, we can conclude that the motion is simple harmonic.

6. What is a Lissajous' figure?

Ans. If two simple harmonic motions at right angles to each other are superposed on each other, the path of the resultant motion is, in general, a closed curve. Such a curve is called a Lissajous' figure.

7. Show that for a body executing SHM, the acceleration leads the velocity by $\frac{\pi}{2}$ and the displacement by π.

Ans. Let us represent the SHM by the following equation:

$$y = A\sin(\omega t + \delta) \quad \text{...(1)}$$

Then, the velocity is given by

$$v = \frac{dy}{dt} = A\,\omega\cos(\omega t + \delta)$$

or,
$$v = A\omega\sin\left(\omega t + \phi + \frac{\pi}{2}\right) \quad \text{...(2)}$$

Now, comparing Eqs. (1) and (2), one can note that velocity leads the displacement by $\frac{\pi}{2}$.

Again, the acceleration is given by

$$f = \frac{dv}{dt} = \frac{d^2y}{dt^2} = -A\omega^2\sin(\omega t + \phi)$$

or,
$$f = A\omega^2\sin(\omega t + \phi + \pi) \quad \text{...(3)}$$

Now, comparing Eqs. (2) and (3), one can see that the acceleration of a particle executing SHM leads the velocity by $\frac{\pi}{2}$. From eqs. (1) and (3) it is evident that the acceleration leads the displacement by π.

8. What are the dimensions of the force constant of a vibrating spring-mass system?

Ans. The force F on the mass of the spring is $F = -kx$, where x is the displacement of the centre of mass from the equilibrium position at any time t.

$\therefore$ the dimension of k is given by

$$[K] = \frac{[F]}{[l]} = \frac{[MLT^{-2}]}{[L]} = [MT^{-2}]$$

i.e., the dimension of the spring constant K is $[K] = [MT^2]$.

9. A hollow sphere is filled with water and used as a pendulum bob. If water trickles out slowly through the hole made at the bottom then how will its time period be affected?

Ans. The time period of a simple pendulum with length l is given by the following equation:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where g is the acceleration due to gravity at the place of the pendulum. When water trickles out slowly through the hole made at the bottom of the hollow sphere, the mass of the bob goes down slowly. Though the time period is independent of the mass m of the pendulum, yet it will vary as $\sqrt{l}$ since due to leaking of water through the hole, the centre of mass of the sphere will keep on changing its positions by varying the effective length. Initially the centre of mass will keep on moving down form the centre of the sphere but after some time it will keep on moving up. And after all the water has trickled away, the centre of mass of the bob will lie again at the centre of the sphere. So, as long as some water will remain in the hollow sphere, its time period will keep on changing. But the initial time period T_i and the final time period T_f will be same (i.e., $T_i = T_f$).

10. How does periodic motion differ from simple harmonic motion?

Ans. Any motion that repeats itself at regular intervals of time on the same path is called periodic motion. On the other hand, the simple harmonic motion is a periodic motion in which the particle traces the same path twice in one period. The locus of the moving particle may or may not be a straight line in case of a periodic motion. But in case of simple harmonic motion, the locus of the moving particle is always a straight line. In case of a simple harmonic motion, the acceleration of the particle is directly proportionnal to the displacement. But in case of a non-simple-harmonic periodic motion, the acceleration is never proportional to the displacement.

Part 2: Descriptive Questions

1. What are the characteristics of SHM? Define time period, frequency, amplitude and phase of a simple harmonic oscillator. Prove the principle of conservation of energy in this case. [WBUT 2001]
2. A cubical box of L cm side and density ρ is floating on water of density d ($\rho < d$). The block is slightly depressed and released. Show that it will execute simple harmonic oscillation. Determine its frequency of oscillation. [WBUT 2001]
3. Establish the differential equation of simple harmonic motion and solve it. [WBUT 2004]

4. Prove that the motion of a simple pendulum (with small oscillation) is a simple harmonic motion. Hence, find the expression of its period of oscillation.
5. Prove that the horizonal motion of a spring-mass system on a frictionless surface is simple harmonic. Hence find the expression for its time period and frequency.
6. Prove that the vertical oscillation of a loaded spring is simple harmonic. And hence find an expression for its time period of oscillation.
7. Two SHMs with the same time period of oscillation differing in phase and amplitude are acting in the same direction on a particle. Deduce expression for the amplitude and phase of the resulting motion. Discuss the special cases while the phase difference is 0, $\pi/2$, and π respectively.
8. What is simple harmonic motion? Show that for a simple harmonic oscillator, the average kinetic energy equals the average potential energy.
9. A simple harmonic oscillator is characterized by $y = \cos \omega t$. Calculate the displacement at which kinetic energy is equal to its potential energy.
10. Calculate the resultant of two simple harmonic oscillations at right angles when their periods are in the ratio of 3:1.
11. Derive the equation of a simple harmonic oscillator form the energy consideration.
12. What oscillates in a simple harmonic electrical oscillator? Can we realize it in practice?

Part 3: Numerical Problems

1. The displacement of a particle executing simple harmonic motion is given by
$$y = \sin kt + \cos kt$$
Find (i) time period, (ii) amplitude of vibration.
2. If $y_1 = 4 \sin (10t + \phi)$ and $y_2 = 5 \cos (10t)$ be the displacements of two particles at time t, then find the phase difference between the velocity of the particles.
3. Two vibrations at right angles to each other are described by the following two equations:
$$x = 5 \cos 3\pi t$$
$$y = 5 \cos \left(6\pi t + \frac{\pi}{4}\right)$$
where x and y are expressed in cm and seconds. Construct the Lissajous' figures.
4. A particle of 100 g mass is held between two rigid supports by two springs of force constants of 8 N/m and 2 N/m. If the particle is displaced along the directions of the lengths of the springs, calculate the frequency of vibration.
5. A body executing simple harmonic motion has an amplitude of 100 cm and a time period of 3 s. Calculate the time taken by the body to travel a distance of $5\sqrt{3}$ cm from its mean position.
6. A particle describes simple harmonic motion in a line 4 cm long. Its velocity, when it passes through the centre of the line, is 16 cm s^{-1}. Find the time period of its oscillation.
7. A point mass m is suspended at the end of a massless wire of length l and cross-sectional area A. If the Young's modulus of elasticity of the wire be Y then obtain the frequency of oscillation for the simple harmonic motion along the vertical line.
8. A particle moves along a straight line with a period of 2.5 seconds and an amplitude of 12 cm. What is the kinetic energy when it is 2 cm away from its position of equilibrium?

CHAPTER

2

Free and Damped Vibrations

2.1 FREE VIBRATIONS

Let us consider a simple pendulum whose bob has been suspended from a rigid support with a string and kept in an evacuated chamber having a transparent window through which the bob can be seen from outside. Let us also assume that the bob is made of a magnetic substance and the string is inflexible and massless. Now, if the bob is displaced from its mean position with the help of a magnet from outside and left itself, then it will keep on oscillating for an indefinite time with a constant amplitude and a constant frequency of vibration. Such a vibration is called a ***free vibration***.

This type of vibration is, however, an ideal case since in reality we cannot have a massless and inflexible string. But this idea helps one develop the concept. So, we can define a free vibration as follows: **The free or undamped or natural vibration is the vibration of a body which is completely free from external forces.** In other words— simple harmonic motions which persist indefinitely without loss of energy and reduction in amplitude are called free or undamped vibrations.

However, observations of the free vibrations of a real physical system reveals that the energy of the vibrator gradually decreases with respect to time and the vibrator eventually comes to rest. For example, the amplitude of a simple pendulum vibrating in air decreases with time and it ultimately stops its vibration. The vibrations of a tuning fork die away with elapsing of time. This happens because of the presence of friction (or damping) in actual physical systems. And the friction always resists motion. In the previous chapter, for the motion of simple harmonic oscillators, we have assumed that their vibrations are free or undamped.

2.2 DIFFERENTIAL EQUATION OF FREE OR UNDAMPED VIBRATIONS

If a particle executes SHM freely, then its kinetic energy for a displacement y, is given by

$$E_k = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2$$

And at the same instant of time, its potential energy is given by $E_p = \frac{1}{2} ky^2$ where k is the restoring force per unit displacement.

So, the total energy (E) at any instant of time is given by,

$$E = E_p + E_k$$

or,
$$E = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2 + \frac{1}{2} ky^2$$

As the concerned oscillator is a free one, its total energy will remain constant,

$\therefore$
$$E = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2 + \frac{1}{2} ky^2 = \text{constant} \qquad ...(2.1)$$

Now, differentiating Eq. (2.1) with respect to time, we get,

$$m\frac{d^2y}{dt^2} + ky = 0 \qquad ...(2.2)$$

or,
$$\frac{d^2y}{dt^2} + \left(\frac{k}{m}\right) y = 0$$

or,
$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad ...(2.3)$$

where $\omega^2 = \left(\frac{k}{m}\right)$

Equation (2.3) is an ideal equation for a free oscillator. In Chapter 1, we considered this equation only.

2.3 DAMPED VIBRATIONS

In real practice, when one causes a pendulum to vibrate in air, there are always frictional forces acting on the system and consequently, the energy of the system gets dissipated in each vibration. The amplitude of vibration decreases continuously with respect to time, and finally the oscillations of the vibrator die away. Such vibrations are known as *free damped* vibrations. The dissipated energy appears as heat either within the system itself or in the surrounding medium. In case of small oscillations, the dissipative forces due to friction (or resistance in case of an LCR circuit) are proportional to the velocity of the vibrator in that instant of time.

Let $\beta \frac{dy}{dt}$ be the dissipative force due to friction. This term is to be introduced in the equation of a free simple harmonic oscillator (SHO), i.e., Eq. (2.2). So the differential equation in case of the free-damped vibrations is given by

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = 0 \qquad ...(2.4)$$

or,
$$\frac{d^2y}{dt^2} + \left(\frac{\beta}{m}\right)\frac{dy}{dt} + \left(\frac{k}{m}\right)y = 0$$

or,
$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = 0 \qquad ...(2.5)$$

where $b = \frac{\beta}{2m}$ and $\omega^2 = \frac{k}{m}$

So, Eq. (2.5) represents the equation of motion of a damped harmonic oscillator.

2.4 SOLUTION OF THE EQUATION OF A DAMPED OSCILLATOR AND ITS ANALYSIS

In order to solve Eq. (2.5), let us put $y = ae^{\alpha t}$ (the trial solution).

Then, $$\frac{dy}{dt} = \alpha ae^{\alpha t} = \alpha y$$

or, $$\frac{d^2y}{dt^2} = \alpha^2 ae^{\alpha t} = \alpha^2 y$$

Now, substituting the values of $\frac{d^2y}{dt^2}$ and $\frac{dy}{dt}$ in Eq. (2.5), we get,

$$\alpha^2 y + 2b\alpha y + \omega^2 y = 0$$

or, $$\alpha^2 + 2b\alpha + \omega^2 = 0$$

or, $$\alpha = -b \pm \sqrt{b^2 - \omega^2}$$

$\therefore$ the general solution of Eq. (2.5) can be written as

$$y = A_1 e^{(-b+\sqrt{b^2-\omega^2})t} + A_2 e^{(-b-\sqrt{b^2-\omega^2})t}$$

or, $$y = e^{-bt}\{A_1 e^{\sqrt{b^2-\omega^2}\,t} + A_2 e^{-\sqrt{b^2-\omega^2}\,t}\} \qquad ...(2.6)$$

where A_1 and A_2 are two constants whose values can be determined from the initial conditions.

Now, depending upon the relative values of b and ω, we can come across the following three cases which deal with three conditions of motion:

Case 1 When $b < \omega$, i.e., the damping force is small, the value of the quantity $\sqrt{b^2 - \omega^2}$ is imaginary and it can be written as

$$\sqrt{b^2-\omega^2} = i\sqrt{\omega^2 - b^2}$$

Hence Eq. (2.6) gets reduced to

$$y = e^{-bt}\left(A_1 e^{i\sqrt{\omega^2-b^2}\cdot t} + A_2 e^{-i\sqrt{\omega^2-b^2}\cdot t}\right)$$

or, $$y = e^{-bt}\{(A_1 + A_2)\cos(\sqrt{\omega^2-b^2})\,t + i(A_1 - A_2)\sin(\sqrt{\omega^2-b^2})t\}$$

or, $$y = e^{-bt}\{A\cos(\sqrt{\omega^2-b^2}.t) + B\sin(\sqrt{\omega^2-b^2}.t)\} \qquad ...(2.7)$$

where $A = A_1 + A_2$ and $B = i(A_1 - A_2)$

Now, let us put, $A = P\cos\theta$ and $B = P\sin\theta$ in Eq. (2.7)

where $P = (\sqrt{A^2 + B^2})$ and $\theta = \tan^{-1}\left(\frac{B}{A}\right)$

So, Eq. (2.7) takes the following form:

$$y = e^{-bt}\{P\cos\theta\cos(\sqrt{\omega^2-b^2})\,t + P\sin\theta\sin(\sqrt{\omega^2-b^2})t\}$$

or, $$y = Pe^{-bt}\cos(\sqrt{\omega^2-b^2}.t - \theta) \qquad ...(2.8)$$

If there is no damping, then $b = 0$ and Eq. (2.8) reduces to

$$y = P\cos(\omega t - \theta) \qquad ...(2.9)$$

which is the solution for the free and undamped SHM.

Now, let us apply the initial conditions in order to evaluate the values of the constants.

Let $y = y_o$ at $t = 0$, so the initial displacement at $t = 0$ is y_o. And the initial velocity at $t = 0$ is given by

$$\dot{y} = \left(\frac{dy}{dt}\right)_{t=0} = v_o \text{ (say)}$$

Putting these values of y, and t in Eq. (2.7), we get

$$\therefore \quad y_o = A \Rightarrow A = y_o$$

Now, from Eq. (2.7), by differentiating with respect to t we get,

$$\dot{y} = -be^{-bt}\{A\cos\sqrt{\omega^2 - b^2}\cdot t + B\sin\sqrt{\omega^2 - b^2}\cdot t\}$$
$$+ e^{-bt}\sqrt{\omega^2 - b^2}\{-A\sin\sqrt{\omega^2 - b^2}\cdot t + B\cos\sqrt{\omega^2 - b^2}\cdot t\}$$

or,
$$(\dot{y})_{t=0} = v_o = -bA + \sqrt{\omega^2 - b^2}\cdot B$$

or,
$$B = \frac{v_o + bA}{\sqrt{\omega^2 - b^2}} = \frac{v_o + b\cdot y_o}{\sqrt{\omega^2 - b^2}}$$

$$\therefore \quad y = e^{-bt}\left\{y_o\cos(\sqrt{\omega^2 - b^2}\cdot t) + \frac{v_o + by_o}{\sqrt{\omega^2 - b^2}}\sin(\sqrt{\omega^2 - b^2})t\right\} \quad \text{...(2.10)}$$

The initial motion of the oscillator can be started by either of the two ways:

(a) by giving initial displacement or

(b) by striking, i.e., by imparting an initial velocity.

Case (a) When $y_o \neq 0$ and $v_o = 0$, we get,

$$y = y_o e^{-bt}\left\{\cos\left(\sqrt{\omega^2 - b^2}\right)\cdot t + \frac{b}{\sqrt{\omega^2 - b^2}}\sin(\sqrt{\omega^2 - b^2})\,t\right\} \quad \text{...(2.11)}$$

Case (b) When $y_o = 0$ and $v_o \neq 0$, we get,

$$y = \frac{v_o e^{-bt}}{\sqrt{\omega^2 - b^2}}\sin(\sqrt{\omega^2 - b^2})t \quad \text{...(2.12)}$$

Now, as $A = P\cos\theta$ and $B = P\sin\theta$ and we know the values of A and B as given above, we get

$$P = \sqrt{A^2 + B^2} = \sqrt{y_o^2 + \left(\frac{v_o + by_o}{\sqrt{\omega^2 - b^2}}\right)^2}$$

or,
$$P = \sqrt{\frac{y_o^2\,\omega^2 + v_o^2 + 2b\,y_o\,v_o}{(\omega^2 - b^2)}}$$

and
$$\theta = \tan^{-1}\left(\frac{by_o + v_o}{y_o\sqrt{\omega^2 - b^2}}\right)$$

If the motion is started by imparting initial displacement, then we obtain,

$$y = y_o e^{-bt}\left\{\cos(\sqrt{\omega^2 - b^2}\cdot t) + \frac{b}{\sqrt{\omega^2 - b^2}}\sin(\sqrt{\omega^2 - b^2}\cdot t)\right\} \quad \text{[Eq. (2.11)]}$$

This equation can be written as

$$y = \left(\frac{\omega y_o}{\sqrt{\omega^2 - b^2}}\right) . e^{-bt} \cos \left\{\sqrt{\omega^2 - b^2} . t - \tan^{-1}\left(\frac{b}{\sqrt{\omega^2 - b^2}}\right)\right\} \qquad ...(2.13)$$

Hints: Let $b = \omega \sin \phi$

$\therefore$ $$\sqrt{\omega^2 - b^2} = \sqrt{\omega^2 - \omega^2 \sin^2 \phi} = \omega \cos \phi$$

$\therefore$ $$\frac{b}{\sqrt{\omega^2 - b^2}} = \frac{\omega \sin \phi}{\omega \cos \phi} = \tan \phi$$

$\therefore$ $$\phi = \tan^{-1} \frac{b}{\sqrt{\omega^2 - b^2}}$$

The motion expressed by Eq. (2.13) represents a damped oscillatory motion whose amplitude $\left(\frac{\omega x_o}{\sqrt{\omega^2 - b^2}} . e^{-bt}\right)$ decreases exponentially with respect to time at a rate determined by the decay constant b and the time period T is given by

$$T = \frac{2\pi}{\sqrt{\omega^2 - b^2}}$$

This damping motion has been shown in Fig. 2.1.

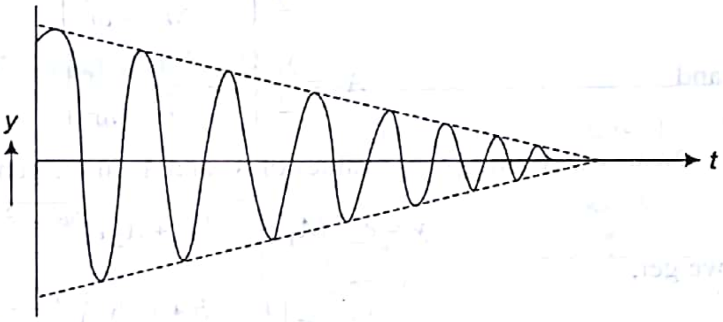


Fig. 2.1 The displacement of a damped oscillator decreases with increase of time and ultimately it dies out.

This decrement of amplitude is called logarithmic decrement of the amplitude of a damped oscillator, because the amplitude is an exponential function of time and is given by

$$A(t) = A_o e^{-bt}$$

where $A(t)$ is the time dependent amplitude and A_o is given by $A_o = \frac{\omega y_o}{\sqrt{\omega^2 - b^2}}$

$\therefore$ $$\ln A(t) = \ln A_o + \ln (e^{-bt})$$

or, $$\ln A(t) = \ln A_o - bt$$

So, if we plot $A(t)$ versus t, we get the graph of Fig. 2.2.

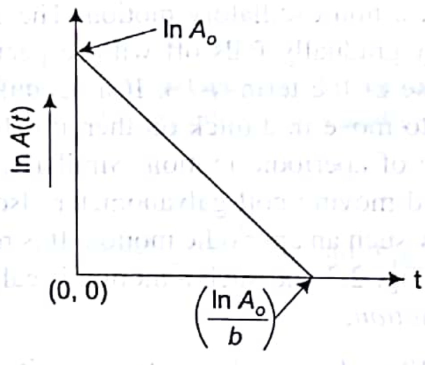


Fig. 2.2 Logarithmic decrement of the amplitude of a damped oscillator with respect to time.

Comparing this damped oscillator motion with the free undamped motion of an oscillator [Eq. (2.9)], we can see that damping force has the following effects on the motion of an oscillator:

(i) The amplitude is not constant but it dies away logarithmically.

(ii) The frequency of oscillation reduces slightly below the natural frequency. The damped frequency is given by $\nu_d = \frac{\sqrt{\omega^2 - b^2}}{2\pi}$ whereas the natural frequency is given by $\nu_n = \frac{\omega}{2\pi}$.

Case 2 When $b > \omega$, i.e., damping force is large, the motion of the oscillator turns to be non-oscillatory. If y_o be the initial displacement and v_o be the initial velocity, then the general solution is given as follows:

$$y = A_1 e^{(-b + \sqrt{b^2 - \omega^2})t} + A_2 e^{(-b - \sqrt{b^2 - \omega^2})t}$$

This equation, at $t = 0$, reduces to

$$y_o = A_1 + A_2 \qquad ...(2.14)$$

or,
$$A_1 + A_2 = y_o$$

Now, $v = \dfrac{dy}{dt} = -be^{-bt}\{A_1 e^{\sqrt{b^2-\omega^2}.t} + A_2 e^{-\sqrt{b^2-\omega^2}.t}\} + \sqrt{b^2 - \omega^2}\,.\,e^{-bt}\{A_1 e^{\sqrt{b^2-\omega^2}.t} - A_2\, e^{-\sqrt{b^2-\omega^2}.t}\}$

At $t = 0$, $v_o = -b\,(A_1 + A_2) + \sqrt{b^2 - \omega^2}\,(A_1 - A_2)$

or,
$$v_o = -by_o + \sqrt{b^2 - \omega^2}\,(A_1 - A_2) \quad \text{[by Equation (2.14)]}$$

or,
$$A_1 - A_2 = \frac{v_o + b_o\, y_o}{\sqrt{b^2 - \omega^2}} \qquad ...(2.15)$$

Now, from Eqs. (2.14) and (2.15) we get,

$$A_1 = \frac{y_o}{2}\left\{1 + \frac{b + v_o/y_o}{\sqrt{b^2 - \omega^2}}\right\}$$

and
$$A_2 = \frac{y_c}{2}\left\{1 - \frac{b + v_o/y_o}{\sqrt{b^2 - \omega^2}}\right\}$$

Now, substituting these values of A_1 and A_2 in the general equation, i.e., in

$$y = e^{-bt}\,(A_1\, e^{\sqrt{b^2-\omega^2}\,t} + A_2\, e^{-\sqrt{b^2-\omega^2}\,t})$$

we get,

$$y = \frac{y_o}{2} e^{-bt}\left\{\left(1 + \frac{b + v_o/y_o}{\sqrt{b^2 - \omega^2}}\right) e^{\sqrt{b^2-\omega^2}\,.\,t} + \left(1 - \frac{b + v_o/y_o}{\sqrt{b^2 - \omega^2}}\right) e^{-\sqrt{b^2-\omega^2}\,.\,t}\right\} \qquad ...(2.16)$$

Equation (2.16) represents an aperiodic motion i.e., a non-oscillatory motion. The displacement y gradually falls off with respect to time because of the term (e^{-bt}). If a pendulum is allowed to move in a thick oil then it exhibits this type of aperiodic motion. Similarly, an over-damped moving coil galvanometer also is able to show such an aperiodic motion. It is represented in Fig. 2.3 and such a motion is called *dead beat motion*.

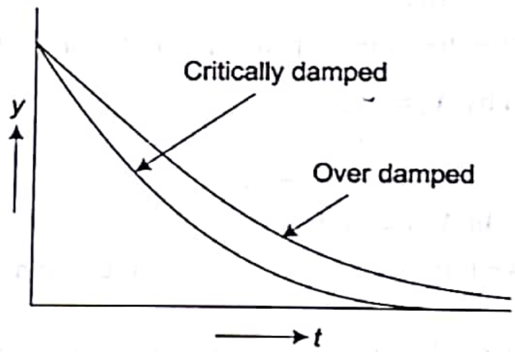


Fig. 2.3 **The overdamped and critically damped motions have been shown. If the daming is slightly less than critical damping then the oscillator will vibrate.**

Case 3 When $b = \omega$, the motion is critical. That is, it is a transitional case when the damped oscillatory motion changes into a dead beat motion. This is known a critical damping and the decay of y is most rapid in this critical case. This motion is illustrated in Fig. 2.3. If the damping is slightly less than this, the damped oscillator starts oscillating.

2.4.1 Solution of Differential Equation in Critical Case

To get the solution, in the critical case, we first take $b \approx \omega$, i.e., $b - \omega \approx 0$ and then finally put $b - \omega = 0$ (i.e., exactly equal to zero).

We can rewrite Eq. (2.16) as follows:

$$y = \frac{y_o}{2} e^{-bt} \left\{ e^{\sqrt{b^2-\omega^2}\,.\,t} + e^{-\sqrt{b^2-\omega^2}\,.\,t} + \frac{b + v_0/y_0}{\sqrt{b^2-\omega^2}} \left(e^{\sqrt{b^2-\omega^2}\,.\,t} - e^{-\sqrt{b^2-\omega^2}\,.\,t}\right) \right\}$$

Now, expanding and neglecting the higher order terms, we get

$$y = \frac{y_o}{2} e^{-bt} \left\{ 1 + (\sqrt{b^2-\omega^2})\, t + 1 - (\sqrt{b^2-\omega^2})\, t + \frac{b + v_o/y_o}{\sqrt{b^2-\omega^2}} (1 + (\sqrt{b^2-\omega^2})t - 1 + (\sqrt{b^2-\omega^2})t) \right\}$$

Finally, upon simplification by putting $b - \omega = 0$, we obtain,

$$y = \frac{y_o}{2} e^{-bt} \{2 + 2\,(b + v_o/y_o)t\}$$

or, $$y = y_o\, e^{-bt} \{1 + (b + v_o/y_o)\, t\} \qquad \ldots(2.17)$$

2.5 ELECTRICAL ANALOGY OF SHM AND DV

Simple harmonic motions (SHM) and damped vibration (DV) can be observed in electrical circuits also like mechanical systems.

2.5.1 SHM in an LC Circuit

Let us consider the following electrical circuit which has been shown in Fig. 2.4. It is an LC circuit.

Figure 2.4 shows an LC circuit which contains one inductor L and one capacitor C connected parallelly. In the equilibrium state the capacitor is uncharged and no current flows in the circuit. As soon as the equilibrium state is disturbed by pressing the key, the capacitor starts charging and the circuit starts oscillating. The voltage across the capacitor is $\frac{q}{c}$ and that across the inductor is $-L\frac{di}{dt} = -L\frac{d^2q}{dt^2}$. The minus sign indicates that the voltage opposes the increase of current. From Kirchhoff's law, the net voltage in the circuit is zero.

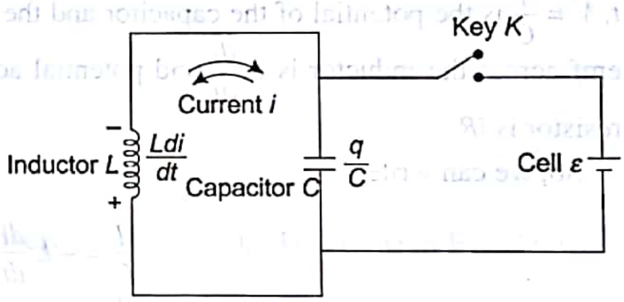


Fig. 2.4 An oscillatory LC circuit, L-inductor, C-Capacitor, K-key and ε-cell.

$$\therefore \qquad -L\frac{d^2q}{dt^2} = \frac{q}{C}$$

or, $$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$$

or, $$\frac{d^2q}{dt^2} + \omega^2 q = 0 \qquad \ldots(2.18)$$

where $\omega^2 = \frac{1}{LC}$

Equation (2.18) is similar to Eq. (2.3) which is the differential equation of SHO.

Thus, in an electrical circuit consisting of an inductor (L) and a capacitor (C) the charge oscillates harmonically with an angular frequency $\omega = \frac{1}{\sqrt{LC}}$ and a period $T = 2\pi\sqrt{LC}$,

$\therefore$ $$q = q_o \cos(\omega t - \phi) \quad \text{[where } q_o \text{ is the initial charge]}$$

or, $$i = \frac{dq}{dt} = -\omega q_o \sin(\omega t - \phi)$$

or, $$i = -i_o \sin(\omega t - \phi)$$

where $i_o = \omega q_o$ is the maximum current in the circuit and i_o is given by $i_o = v_o\sqrt{\frac{C}{L}}$

2.5.2 DV in an LCR Circuit

Let us consider the following electrical circuit which has been shown in Fig. 2.5. It is an LCR circuit. It is capable of showing damped oscillations. When $R = 0$, the oscillations of the circuit are undamped with angular frequency $\omega_o = \frac{1}{\sqrt{LC}}$. The resistance generates the resistive (or dissipative) force in the circuit. As soon as the key K is pressed, the capacitor gets charged by the battery and on release of the key, the battery is thrown out of the circuit and the capacitor starts discharging. At time t, $V = \frac{q}{C}$ is the potential of the capacitor and the induced emf across the inductor is $L\frac{di}{dt}$ and potential across the resistor is iR.

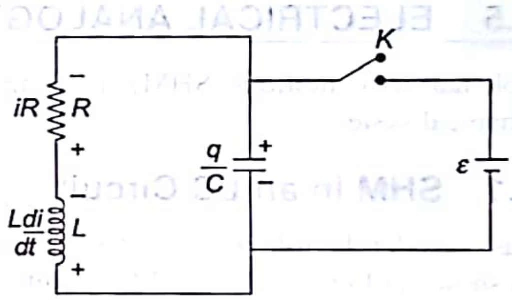


Fig. 2.5 An LCR circuit which shows damped vibration. R-resistance, L-inductor, C-Capacitor ε-source, i-current, K-key.

So, we can write,

$$\frac{q}{C} = -L\frac{di}{dt} - iR$$

or, $$L\frac{di}{dt} + iR + \frac{q}{C} = 0$$

or, $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

or, $$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0$$

or, $$\frac{d^2q}{dt^2} + 2b\frac{dq}{dt} + \omega^2 q = 0 \quad \text{...(2.19)}$$

where $\frac{R}{L} = 2b$ and $\frac{1}{LC} = \omega^2$

Equation (2.19) is similar to Eq. (2.5) which is the equation for a damped vibration.

So, the LCR circuit represents a damped vibrator.

Worked-out Examples

Example 2.1 Show that $y = (A + Bt)\, e^{-\frac{\beta t}{2m}}$ is the solution of the following differential equation for damped vibration:

$$\frac{d^2y}{dt^2} + \frac{\beta}{m}\frac{dy}{dt} + \frac{k}{m}y = 0$$

for the critically damped oscillations where m is the mass of the vibrating system, β is the resistive force per unit velocity and k is the restoring force per unit displacement.

Sol. If the vibrating system gets displaced along the y-axis then the differential equation of a damped harmonic motion is given by

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = 0 \qquad \ldots(1)$$

where $2b = \frac{\beta}{m}$, $\omega^2 = \frac{k}{m}$

Now, given,

$$y = (A + Bt)\, e^{-\frac{\beta t}{2m}} = (A + Bt)\, e^{-bt}$$

or,
$$y = (A + Bt)\, e^{-bt} \qquad \ldots(2)$$

$\therefore$
$$\frac{dy}{dt} = -b(A + Bt)\, e^{-bt} + Be^{-bt} \qquad \ldots(3)$$

and
$$\frac{d^2y}{dt^2} = b^2 (A + Bt)\, e^{-bt} - 2b\, Be^{-bt} \qquad \ldots(4)$$

Now, substituting the values of y, $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ respectively from Eqs. (2), (3) and (4) in Eq. (1), we get,

$$b^2 (A + Bt)\, e^{-bt} - 2bBe^{-bt} - 2b^2 (A + Bt)\, e^{-bt} + 2bBe^{-bt} + \omega^2 (A + Bt)\, e^{-bt} = 0$$

or,
$$(\omega^2 - b^2)(A + Bt)\, e^{-bt} = 0$$

or,
$$(\omega^2 - b^2)\, y = 0$$

But y cannot be zero for all times,

$\therefore$
$$\omega^2 - b^2 = 0$$

or,
$$\omega^2 = b^2$$

This is the condition for critical damping of a vibration. Hence, the given expression $y = (A + Bt)\, e^{-\frac{\beta t}{2m}}$ is the solution of the given differential equation for the critically damped oscillation.

Example 2.2 What is decay constant (or relaxation time)? How does it vary with damping coefficient? [WBUT 08]

Sol. The relaxation time of a damping oscillator is defined as the time in which the amplitude of a damped oscillator decays to $\frac{1}{e}$ time of its initial amplitude.

If A_o and A_t be the amplitudes of the oscillator at times $t = 0$ and at time $t = t$ respectively then we can write,

$$\frac{A_t}{A_o} = \frac{1}{e}$$

But $\quad e = 2.717 \quad \Rightarrow \quad \frac{1}{e} = 0.368$

$\therefore \quad A_t = 0.368\, A_o$

Hence, the higher is the decay constant (or relaxation time, τ_r), the slower is the rate of decay of the amplitude. So, in such a case the rate of dissipation of energy of the damped oscillator also will be slow.

For a damping system, the amplitude varies with time exponentially.

So, we can write,

$$A_t = A_o\, e^{-bt}$$

where $b = \dfrac{\beta}{2m}$ and β is the damping coefficient, i.e., damping force per unit velocity of the oscillator.

Now, if time t is equal to the relaxation time τ_r (i.e., $t = \tau_r$), then, we get,

$$A_t = A_o\, e^{-b\tau_r} = \frac{1}{e} A_o \quad \text{[by definition]}$$

or, $\quad e^{-b\tau_r} = e^{-1}$

or, $\quad b\tau_r = 1 \quad \Rightarrow \quad \tau_r = \dfrac{1}{b}$

or, $\quad \tau_r = \dfrac{2m}{\beta}$

So, the relaxation time (τ_r) of the damped oscillator varies inversely as the damping coefficient (β).

Example 2.3 Show that in case of damped oscillation with a small amount of damping, the corresponding time-period of oscillation is higher than the time period of free-oscillation.

Sol. In case of small damping, the angular frequency $\omega_d = \sqrt{\omega^2 - b^2}$ where ω is the angular frequency of a similar undamped oscillator and $b = \dfrac{k'}{2m}$, k' being damping coefficient.

$\therefore$ the time period T_d of the damped oscillator is given by

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{\omega^2 - b^2}}$$

whereas the time period of an undamped oscillator is given by

$$T = \frac{2\pi}{\omega}$$

From the values of T_d and T

$$T_d > T \quad [\because \sqrt{\omega^2 - b^2} < \omega]$$

Example 2.4 Show that in case of damped vibrator, the rate of loss of energy of the vibrator is equal to the rate of work done by the vibrator against the resisting force.

Sol. For a damped vibrator, the equation of motion is given by,

$$m\frac{d^2y}{dt^2}+k'\frac{dy}{dt}+ky=0$$

or, $$m\frac{d^2y}{dt^2}+ky=-k'\frac{dy}{dt}=\text{resisting force}=F_{res}\ (\text{say})$$

The total energy of the vibrator is given by

$$E=E_k+E_p$$

or, $$E=\frac{1}{2}m\left(\frac{dy}{dt}\right)^2+\frac{1}{2}ky^2$$

$\therefore$ $$\frac{dE}{dt}=\frac{m}{2}.2.\left(\frac{dy}{dt}\right).\frac{d^2y}{dt^2}+\frac{k}{2}.2y.\frac{dy}{dt}$$

or, $$\frac{dE}{dt}=\left(m\frac{d^2y}{dt^2}+ky\right)\frac{dy}{dt}$$

$$=\left(-k'\frac{dy}{dt}\right)\left(\frac{dy}{dt}\right)=F_{res}\frac{dy}{dt}$$

or, $$\frac{dE}{dt}=\frac{d}{dt}(F_{res}\,.\,y)=\frac{d}{dt}(W)$$

where W is the work done against the resistive force.

So, $$\frac{dE}{dt}=\frac{dW}{dt}$$

i.e., the rate of loss of energy of the damped vibrator is equal to the rate of work done against the resisting force.

Example 2.5 The amplitude of an oscillator of 200 cps frequency falls to (1/10)th of its initial value after 2000 cycles. Calculate its relaxation time, quality factor and time in which its energy falls to (1/10)th of its initial value. [WBUT 2008]

Sol. Let Q be the quality factor of the oscillator, ν be its frequency, m be its mass and β be the damping force for per unit velocity.

The damping constant b is given by

$$b=\frac{\beta}{2m}$$

$\therefore$ the amplitude at any time t is given by,

$$A(t)=A_o\,e^{-bt}$$

where A_o is the initial amplitude.

$\therefore$ $$\frac{A(t)}{A_o}=e^{-bt} \quad \text{...(1)}$$

The frequency $\nu=200\ \text{s}^{-1}$

$\therefore$ time period $T=\frac{1}{\nu}=\frac{1}{200}$ s

Time taken for 2000 cycles is given by

$$t=10\,T=\frac{10}{200}\text{ s}=\frac{1}{20}\text{ s}$$

Scanned by CamScanner

∴ from Eq. (1), we can write

$$e^{-bt} = \frac{A(t)}{A_o} = \frac{1}{10} = 10^{-1}$$

or, $\log_{10}(e^{-bt}) = -1$

or, $-bt \log_{10}^{e} = -1$

or, $$b = \frac{1}{t \times \log_{10}^{e}} = \frac{1}{\frac{1}{20} \times \log_{10}^{e}}$$

or, $$b = \frac{20}{\log_{10}^{e}} = 20 \ln 10$$

or, $$b = 20 \times 2.3 = 4.6$$

∴ the quality factor Q is given by

$$Q = \frac{\omega}{2b} = \frac{2\pi \times 200}{2 \times 4.6} = 136.6$$

The relaxation time τ_r is given by,

$$\tau_r = \frac{1}{b}$$

or, $$\tau_r = \frac{1}{4.6}$$

or, $$\tau_r = 0.217 \text{ s}$$

We know, for a damping system, that the energy at time t is given by

$$E_t = E_o e^{-2bt}$$

Let E_i and E_f be energies at $t = 0$ and $t = t_f$.

∴ $$E_f = E_i e^{-2bt_f} \quad [\because E_i = E_0 e^{-2b.0} = E_o]$$

or, $$e^{-2bt_f} = \frac{E_f}{E_i} = \frac{1}{10}$$

or, $$e^{-2bt_f} = 10^{-1}$$

or, $$-2bt_f = -\ln 10$$

or, $$2bt_f = \ln 10$$

or, $$t_f = \frac{1}{2b} \times \ln 10$$

or, $$t_f = \frac{1}{2 \times 4.6} \times 2.3$$

$$t_f = \frac{1}{4} = 0.25 \text{ s}$$

***Note:* *Energy of a damped oscillator*:**

The total energy of any oscillator is given by

$$E = E_k + E_p$$

where E is the total energy, E_k is the kinetic energy and E_p is the potential energy.

$$\therefore \quad E = \frac{1}{2}mv^2 + \frac{1}{2}ky^2 \quad ...(1)$$

where k is the restoring force per unit change of displacement y.

If the oscillator is a damped oscillator, then its displacement is given by

$$y = ae^{-bt}\sin(\omega' t + \delta) \quad ...(2)$$

where $\omega' = \sqrt{\omega^2 - b^2}$

$$\therefore \quad v = \frac{dy}{dt} = ae^{-bt}(\omega')\cos(\omega' t + \delta) \quad [\because b \text{ is very small}] \quad ...(3)$$

If the oscillator is excited by giving a velocity v_o suddenly in the mean position then at $t = 0$, $\frac{dy}{dt} = v_o$ and $\delta = 0$

$$\therefore \quad v_o = a\omega' \Rightarrow a = v_o/\omega'$$

$$\therefore \quad y = \frac{v_o}{\omega'}e^{-bt}\sin(\omega' t) \quad ...(4)$$

and $$v = v_o e^{-bt}\cos(\omega' t) - \frac{v_o b}{\omega'}e^{-bt}\sin(\omega' t)$$

$$\therefore \quad v \approx v_o e^{-bt}\cos(\omega' t) \quad [\because b \text{ is very small}] \quad ...(5)$$

$\therefore$ Eq. (1) can be written as

$$E = \frac{1}{2}mv_o^2 e^{-2bt}\cos^2(\omega' t) + \frac{kv_o^2}{2\omega'^2}e^{-2bt}\sin^2(\omega' t) \quad [\text{using Eqs. (3) and (4)}]$$

or, $$E = \frac{1}{2}mv_o^2 e^{-2bt}\cos^2\omega' t + \frac{1}{2}mv^2\left(\frac{\omega}{\omega'}\right)^2 e^{-2bt}\sin^2\omega' t \quad [\because k = m\omega^2]$$

or, $$E = \frac{1}{2}mv_o^2 e^{-2bt}\left[\cos^2\omega' t + \frac{\omega^2}{\omega'^2}.\sin^2\omega' t\right]$$

In case of low damping

$$\omega' = \sqrt{\omega^2 - b^2} \approx \omega = \text{natural angular frequency}$$

$$\therefore \quad E = \frac{1}{2}mv_o^2 e^{-2bt}[\cos^2\omega' t + \sin^2\omega' t]$$

or, $$E = \frac{1}{2}mv_o^2 e^{-2bt}$$

Initially, at $t = 0$, $E = \frac{1}{2}mv_o^2 = E_o$ (say)

$$\therefore \quad E = E_o e^{-2bt} \quad ...(6)$$

i.e., in case of small damping, the energy decays exponentially as given by Eq. (6).

Example 2.6 In damped harmonic motion, calculate the time in which the energy of the system falls to e^{-1} times of its initial value. [WBUT 2007].

Sol. The energy at time t in damping motion is given by

$$E(t) = E_o e^{-2bt}$$

Let E_f be the energy at time t_f when $E_f/E_i = \frac{1}{e}$ where $E_i = E_o$

$$\therefore \quad \frac{E_f}{E_i} = e^{-2bt_f} = \frac{1}{e}$$

or,
$$e^{-2bt_f} = e^{-1}$$

$$\therefore \quad 2bt_f = 1$$

$$\therefore \quad t_f = \frac{1}{2b} \qquad \ldots(1)$$

Here, b is the damping constant. So, if value of b is given then value of time t_f can be numerically calculated.

Example 2.7 Write down the differential equation of a series LCR circuit driven by a sinusoidal voltage. Identify the natural frequency of this circuit. Find out the condition that this circuit will show an oscillatory decay and find out the relaxation time. [WBUT 2007].

Sol. Refer to subsection 2.5.2 for the first part.

The natural frequency of the circuit is given by

$$\omega = \frac{1}{\sqrt{LC}} = 2\pi\nu$$

where ω is the angular frequency and ν is the frequency.

$$\therefore \quad \nu = \frac{1}{2\pi\sqrt{LC}}$$

The damping constant b is given by

$$2b = \frac{R}{L} \quad \Rightarrow \quad b = \frac{R}{2L}$$

The condition for the circuit to be oscillatory damping is that b must be small. So the circuit should have low resistance or high inductance.

The relaxation time τ_r is given by

$$\tau_r = \frac{1}{b} = \frac{2L}{R}$$

Example 2.8 The damped frequency of vibration of a body is 200 Hz. The amplitude of vibration becomes $\frac{1}{e}$ of the initial amplitude after 1 second. Calculate the frequency of free vibration.

Sol. For vibration to occur, the damping frequency must be low.

$\therefore$ the amplitude at any time t is given by

$$A(t) = A_o e^{-bt}$$

And the amplitude after 1 second is

$$A(t+1) = A_o\, e^{-b(t+1)}$$

$\therefore$ according to the given condition,

$$A(t+1) = \frac{1}{e} A(t)$$

or,
$$A_o e^{-b(t+1)} = \frac{1}{e} A_o\, e^{-bt}$$

or, $e^{-b(t+1)} = e^{-bt-1}$

or, $bt + b = bt + 1$

$\therefore$ $b = 1$

For low damping, the displacement can be written as

$$y = A_o e^{-bt} (\sin (\sqrt{\omega^2 - b^2})\, t - \delta)$$

So, we can write

$\sqrt{\omega^2 - b^2} = 2\pi \times \nu_d$ where ν_d is the frequency of damped oscillation.

or, $\sqrt{\omega^2 - b^2} = 2\pi \times 200$

or, $\omega^2 - b^2 = (400\,\pi)^2$

or, $\omega^2 = (400\,\pi)^2 + 1^2$ $[\because b = 1]$

or, $\omega = \sqrt{1579201}$

$\therefore$ the frequency of free vibration is given by

$$\nu = \frac{\omega}{2\pi} = \sqrt{\frac{1579201}{2\pi}} \approx 200 \text{ Hz}$$

Example 2.9 In a series LCR circuit driven by a *dc* source of emf, the values of $L = 1$ mH, $C = 5$ μF and $R = 0.5$ ohm. Calculate the frequency, the relaxation time and Q-factor of the circuit.

Sol. The angular frequency ω' of the driven oscillator is given by

$$\omega' = \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)^{\frac{1}{2}}$$

where $L = 1 \text{ mH} = 10^{-3}$ H,

$C = 5\ \mu\text{F} = 5 \times 10^{-6}$ F

and $R = 0.5$ ohm

Putting the above-mentioned values in the expression for ω', we get

$$\omega' = \left[\frac{1}{10^{-3} \times 5 \times 10^{-6}} - \frac{(0.5)^2}{4 \times 10^{-6}}\right]^{\frac{1}{2}}$$

$\therefore$ $\omega' = 1.414 \times 10^4$ rad s^{-1}

Again, $\omega' = 2\pi \nu'$ where ν' is the frequency.

$\therefore$ $$\nu' = \frac{\omega'}{2\pi} = \frac{1.414 \times 10^4}{2\pi} = 0.225 \times 10^4 \text{ Hz}$$

Now, the relaxation time τ_r is given by

$$\tau_r = \frac{1}{b} = \frac{2L}{R} = 4 \times 10^{-3} \text{ s}$$

and the quality factor Q is given by

$$Q = \frac{\omega'}{2b} = \frac{\omega'}{R/L} = \frac{\omega' L}{R}$$

or, $$Q = \frac{1.414 \times 10^4 \times 10^{-3}}{0.5} = \frac{1.414 \times 100}{5}$$

$\therefore$ $Q = 28.3$

Example 2.10 A mechanical oscillator has initial energy $E_o = 50$ J having a damping coefficient $b = 1\ s^{-1}$. Calculate the time required to decrease to $E_t = E_o/e$.

Sol. If E_o and E_t be the energies of the oscillator at $t = 0$ and $t = t$ respectively then,

$$E_t = E_o\, e^{-2bt}$$

or,
$$\frac{E_t}{E_o} = e^{-2bt}$$

But according to the condition given,

$$E_t = E_o/e \quad \Rightarrow \frac{E_t}{E_o} = \frac{1}{e}$$

$\therefore$
$$e^{-2bt} = e^{-1}$$

or,
$$2bt = 1$$

or,
$$2 \times 1 \times t = 1 \quad [\because b = 1\mathrm{s}^{-1}]$$

or,
$$t = 0.5 \text{ s}$$

$\therefore$ it will take 0.5 second for the energy to decrease to E_o/e.

Review Exercises

Part 1: Multiple Choice Questions

1. ω is the natural frequency of an oscillator and b is the damping factor ($b = \beta/(2m)$) where β = force per unit velocity, the quality factor of the oscillator is given by

   (a) ω/b (b) $\omega/(2b)$ (c) $\frac{2\omega}{b}$ (d) $\frac{b}{\omega}$

2. Relaxation time is the time in which the amplitude A_i of the damped oscillator falls to

   (a) $\frac{A_i}{3}$ (b) $\frac{A_i}{b}$ (c) $\frac{A_i}{e}$ (d) Aie

3. The quality factor of a series LCR circuit is

   (a) $\frac{1}{R\sqrt{LC}}$ (b) $\frac{1}{R}\sqrt{\frac{L}{C}}$ (c) $\sqrt{\frac{RL}{C}}$ (d) $\sqrt{RLC}$

4. The relaxation time is defined as the time during which the amplitude of a damped oscillator

   (a) grows to e times the initial value
   (b) decays to $1/e$ times the initial value
   (c) grows to e^2 times the initial value
   (d) decays to $1/e^2$ times the initial value

5. Which of the following relation is true for logarithmic decrement?

   (a) $\lambda = b^2 T$ (b) $\lambda \sqrt{bT}$ (c) $\lambda = bT$ (d) $\lambda = bT^2$

6. The quality factor of an electrical oscillator is

   (a) $\frac{\omega R}{L}$ (b) $\frac{LR}{\omega}$ (c) $\frac{L\omega}{R}$ (d) $\frac{L}{R\omega}$

7. For a small value of the damping constant, the quality factor
   (a) decreases (b) remains constant (c) increases (d) none of these
8. Decay of the oscillations caused by frictional resistive forces is called
   (a) rarefactions (b) forced vibration (c) damping (d) decay of oscillation
9. When the frequency of the driving force is equal to that of a driven vibrator, the phase difference between the duo is
   (a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{-\pi}{2}$ (d) π
10. Select the correct statement from the following:
    (a) The higher the damping, the higher the quality factor.
    (b) The lower the damping, the higher the quality factor.
    (c) Damping is not related to quality factor.
    (d) None of these.
11. In case of critical damping, the motion of a system is
    (a) oscillatory (b) vibratory (c) harmonic (d) non-oscillatory
12. If F_o be the amplitude of the driving force and k be the restoring force per unit displacement then the amplitude of the forced oscillator for $\omega = 0$ is
    (a) $\frac{k}{F_o}$ (b) $\frac{F_o}{k}$ (c) kF_o (d) $\frac{1}{kF_o}$
13. The sharpness of a velocity resonance curve is high if the damping constant
    (a) b is medium (b) b is small (c) $b \to \infty$ (d) none of these
14. The amplitude of a damped oscillator with a damping factor b varies as
    (a) e^{2bt} (b) e^{bt} (c) e^{-bt} (d) $e^{\frac{2}{bt}}$
15. If τ_r be the relaxation time and b be the damping constant then
    (a) $\tau_r = \frac{2}{b}$ (b) $\tau_r = \frac{1}{b}$ (c) $\tau_r = b$ (d) $\tau_r b = \text{constant}$

[**Ans.** 1(b), 2(c), 3(b), 4(b), 5(c), 6(c), 7(c), 8(c), 9(a), 10 (b), 11(d), 12 (b), 13(a), 14(c), 15(b)]

Short Questions with Answers

1. **What is the physical significance of the logarithmic decrement of a damped oscillatory system?**

Ans. The logarithmic decrement in case of a damped oscillator gives the measure of the rate at which the amplitude of oscillation decays. Since the logarithmic decrement is the damping factor multiplied by time period, it gives a method of evaluation of the damping coefficient.

2. **Define free vibration. Can a real vibrator vibrate completely freely. If not why?**

Ans. A free vibrator is such a vibrator which keeps on vibrating with the same amplitude for an indefinite time after it is set into vibration. The concept of a free vibrator is an ideal case. In reality, no vibrator can vibrate for a long time with the same amplitude. Any real vibrator faces some resisting forces due to which its amplitude of vibration keeps on decreasing slowly and it ceases to vibrate after some time.

3. What are the sources of damping force?

Ans. The various sources of damping are recorded below:

(i) The loss of energy by radiation of an oscillator

(ii) The loss of energy by mechanical friction (i.e., viscosity, etc.)

(iii) The transfer of heat energy from a layer at higher temperature to a layer at lower temperature

(iv) The intermolecular exchange of energy

4. Define logarithmic decrement. And derive a relation for it.

Ans. Let A_o and A_t be the amplitudes of a damped oscillator respectively at time $t = 0$ and $t = t$. A_t is given by

$$A_t = A_o e^{-bt}$$

where b is the damping constant and is given by $b = \frac{\beta}{2m}$ where β is the damping force per unit velocity and m is the mass of the oscillator. If the oscillator starts from its mean position, then at $t = T/4$ (i.e., one fourth of its time period) it goes to its maximum displacement. Let this displacement be denoted by A_1. Then we have

$$A_1 = A_o e^{-\frac{bT}{4}}$$

The oscillator goes to its maximum displacement (A_2) on the same side after a time $\left(T + \frac{T}{4}\right)$, where

$$A_2 = A_o e^{-b\left(T + \frac{T}{4}\right)}$$

Similarly, the successive amplitudes $A_3, A_4, \ldots A_n$, on the same side are given by

$$A_3 = A_o e^{-b\left(2T + \frac{T}{4}\right)}$$

$$A_4 = A_o e^{-b\left(3T + \frac{T}{4}\right)}$$

$$\ldots$$

$$A_n = A_o e^{-b\left(\overline{n-1} \times T + \frac{T}{4}\right)}$$

$$\therefore \quad \frac{A_1}{A_2} = \frac{A_2}{A_3} = \frac{A_3}{A_4} = \ldots = \frac{A_{n-1}}{A_n}$$

$$= e^{+bT} = d \text{ (say)}$$

d being a constant

This constant d is known as decrement.

or, $\quad \ln\left(\frac{A_{n-1}}{A_n}\right) = bT$

or, $\quad \ln\left(\frac{A_i}{A_{i+1}}\right) = bT = \lambda \text{ (say)}$

The constant λ is called logarithmic decrement.

So, λ can be defined as the natural logarithm of the ratio of any amplitude to the next amplitude on the same side of the mean position of a damped oscillation separated by one full time period.

5. What is critical damping?

Ans. Critical damping is a border line case. If the damping constant of such an oscillating system is increased a bit, it will stop oscillating and if the damping constant is decreased a bit, it will start oscillating. In its critical damping stage, the system neither behaves as an oscillator nor as a non-oscillator. And in case of critical damping the amplitude dies very quickly to zero.

6. Distinguish between free and damped oscillation.

Ans.

Free oscillation	*Damped oscillation*
(i) It persists indefinitely without loss of energy.	(i) It cannot persist for long time and the energy decreases gradually.
(ii) It is an ideal case; usually it is not seen in nature.	(ii) It is a real case. It is very frequently seen.
(iii) Once started it continues for ever with same amplitude and frequency.	(iii) Its amplitude and frequency dies with time.
(iv) It is a conservative system in which the energy is conserved.	(iv) It is a non-conservative system so its energy is not conserved.
(v) *Example:* Oscillation of a simple pendulum in absolute vacuum	(v) *Example:* Oscillation of a simple pendulum in air.

Part 2: Descriptive Questions

1. What is meant by damping? Show that for small damping, the average fractional loss of energy of a particle executing damped simple harmonic motion in one cycle is four times of the logarithmic decrement.
2. What are the conditions for overdamped, critically damped and underdamped motions? Write the displacement–time relationship in each case.
3. Write down the differential equation for a damped oscillations of a system from the energy principle. Solve this differential equation.
4. What are the similarities and dissimilarities of critically damped and overdamped motion? Show that the ratio of the successive amplitudes of damped oscillatory motion is constant.
5. Define the following terms in connection with the damped simple harmonic motion of a particle: (a) decay constant, (b) logarithmic decrement, and (c) quality factor.
6. (a) What is meant by critical damping? [WBUT 2004]

   (b) Establish the differential equation of a damped harmonic motion and explain the different terms in the expression of the equation.
7. (a) Prove that the vertical motion of a loaded spring immersed in water is damped simple harmonic and hence find out an expression for the time period of its oscillation if it is underdamped.
8. Write down the differential equation of an *LCR* circuit. Make a comparison between mechanical parameters and electrical parameters in relation to vibration.

Part 3: Numerical Problems

1. A mass of 1 kg is suspended from a spring with a stiffness constant of 24 Nm^{-1}. If the undamped frequency is $\frac{2}{\sqrt{3}}$ times the damped frequency, calculate the damping factor.

2. A damped vibrating system, starting from rest, reaches a first amplitude of 500 mm which reduces to 50 mm in that direction after 100 oscillations, each of period 2.3 seconds. Find the damping constant and correction for the first displacement for damping.
3. A simple pendulum has a period of 1 second and an amplitude of 10 mm. After 10 complete oscillations, its amplitude is reduced to 5 mm. What is the relaxation time of the pendulum and quality factor?
4. The motion of a particle of mass $m = 0.1$ kg, subjected to a restoring force of 0.1 Nm^{-1}, is critically damped. While at rest, its motion is started by an initial velocity of 0.5 m s^{-1}. Find the maximum displacement of the particle.
5. A body of 10 g mass executes one-dimensional motion. It is acted upon by a restoring force per unit displacement of 10 dyne/cm and a restoring force per unit velocity of 2 dyne/(cms^{-1}). Find the value of the resisting force which will make the motion critically damping. Also, find the value of mass for which the given forces will make the motion critically damping.
6. The motion of an oscillatory system of 100 g mass, subjected to a restoring force of 100 dyne/cm, is critically damped. When it remains at rest, its motion is started by an initial velocity of 50 cm/s. Find the maximum displacement of the oscillator.
7. A vibrating system of 1g mass is displaced from its mean position of rest and released. If it is acted upon by a restoring force of 5 dyne/cm and a damping force of 1 dyne/(cm s^{-1}), check if the motion is a periodic or oscillatory. If the initial displacement of the system is 5 cm, what would be the displacement after a time duration of 5 seconds?

CHAPTER

3

Forced Vibrations

3.1 INTRODUCTION

When a vibrating system is acted upon by an external periodic force, it is assumed to be in a state of forced vibrations (or forced oscillations). Initially, the body tends to vibrate with its own natural frequency, while the applied force tries to make the body vibrate with the frequency of itself (i.e., the frequency of the periodic force). The natural vibration of the forced body dies away quickly due to the influence of the applied force and ultimately it vibrates with a frequency which is same as that of the applied force. Such vibrations are known as **forced vibrations.**

Examples of forced vibrations are (i) vibration of the couple of a tuning fork and a table, and (ii) the coupled vibrations of soldiers and a bridge when they (the soldiers) march on the bridge.

When the tuning fork forces the table to vibrate in the fork's own natural frequency, the table vibrates with small amplitude. But if both the natural frequencies are equal to each other, the table vibrates with a large amplitude. **This phenomenon is called resonance**. The vibration of the tuning fork continues for a longer time if it is not put in contact with the table, whereas if it is held upon the table, the vibration dies out quickly. The energy of the tuning fork is drained very quickly to set the table as well as the larger volume of the air in contact of it to vibrate.

If the forced system cannot modify (by its reaction) the forcing system (i.e., the energy of the forcing system remains unchanged) then the vibration is called **forced vibration.** And if the mass of the forcing system is very large (as compared to that of the forced system) then only forced vibration is possible.

On the other hand, when the forcing system is modified by the forced system it becomes a coupled system and the oscillation is called **coupled oscillation.**

3.2 SYMPATHETIC VIBRATION OR RESONANCE

If the frequencies of the forcing and the forced systems become equal then these systems vibrate with a large amplitude of vibration. The aforesaid phenomenon is called resonance or sympathetic vibration.

Let us try to understand the forced vibration with the help of an example, given as follows:

3.2.1 Barton's Experiment of Forced Vibration

Barton's experimental set-up has been shown in Fig. 3.1. A thick and several feet long cord AB has been stretched between two rigid supports at A and B, but not stretched too much tightly. Six pendulums of different lengths are hanging from it. The pendulums CD and EF have the same length. CD has a heavy brass bob while the other bobs, including EF, consist of light small paper cones so that these pendulums are heavily damped when they vibrate. As the lengths of CD and EF are equal and they are intermediate between the lengths of the shortest and the longest of the light pendulums, whenever the heavy pendulum CD is set into vibration, its motion is communicated through the thick cord and the other pendulums vibrate with the same frequency as that of the pendulum CD. But as they are of different lengths (excluding EF) and hence of different natural frequencies, they show different amplitudes of vibration. Initially, the amplitude of EF is equal to that of CD, but after some time it becomes small as compared to that of CD as it is highly damped. The pendulum EF picks up the vibration immediately. Now brass rings are slipped on to each of the paper cones for making them heavier. As their masses considerably increase while the resistive forces remain unaltered, the damping factor will be much reduced. This time initial natural vibration will last for a longer time to reach the steady-state vibration and also the resonance effect will be much pronounced.

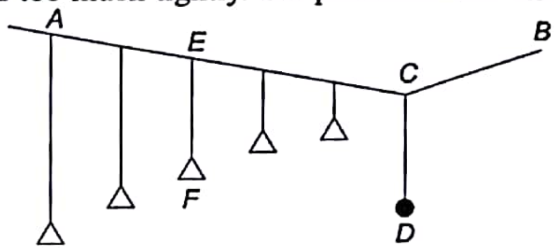


Fig. 3.1 Barton's experiment of forced vibrations.

3.3 ANALYSIS OF FORCED VIBRATION

Let us represent the external simple harmonic force acting on a particle of mass m by $F \cos \omega' t$ where F is its magnitude and ω' is its angular velocity. Let y be the displacement of the particle executing damping oscillatory motion. Let us assume that at $t = 0$ it is at rest at the mean position. The forces acting on the particle are

(i) The restoring or restitution force F_r where $F_r \propto y$

(ii) The retarding or resistive force $F_{res} \propto \dfrac{dy}{dt}$

(iii) The internal force F_i where $F_i = m\dfrac{d^2y}{dt^2}$

(iv) The external periodic force $F_p = F \cos \omega' t$

So the equation of motion of the particle is given by

$$m\frac{d^2y}{dt^2} = F_p - F_r - F_{res}$$

or,
$$m\frac{d^2y}{dt^2} = F\cos\omega' t - ky - k'\frac{dy}{dt}$$

or,
$$\frac{d^2y}{dt^2} + \frac{k'}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{F}{m}\cos\omega' t$$

or,
$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = f\cos\omega' t \qquad ...(3.1)$$

where $2b = \frac{k'}{m}$, $\omega^2 = \frac{k}{m}$ and $f = \frac{F}{m}$

In the steady-state condition, the system will oscillate with the same frequency as that of the external periodic force $F \cos \omega' t$.

Hence, from the physical point of view, let us consider the solution of Eq. (3.1) to be of the following form:

$$y = A \cos (\omega' t - \alpha) \quad \text{...(3.2)}$$

Equation (3.2) represents an SHM of amplitude A and frequency $\nu' = \frac{\omega'}{2\pi}$ and phase lagging behind that of the forcing system by an angle α.

Now, differentiating Eq. (3.2) with respect to t, we get,

$$\frac{dy}{dt} = -A\omega' \sin (\omega' t - \alpha)$$

Differentiating again with respect to t, we get,

$$\frac{d^2y}{dt^2} = -A\omega'^2 \cos (\omega' t - \alpha)$$

Now, substituting the values of $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ in Eq. (3.1), we obtain,

$$-A\omega'^2 \cos (\omega' t - \alpha) - 2\,b\,A\omega' \sin (\omega' t - \alpha) + \omega^2 A \cos (\omega' t - \alpha) = f \cos (\omega' t)$$

or, $-A\omega'^2 \cos (\omega' t - \alpha) - 2\,b\,A\omega' \sin (\omega' t - \alpha) + \omega^2 A \cos (\omega' t - \alpha) = f \cos (\overline{\omega' t - \alpha + \alpha})$

or, $-A\omega'^2 \cos (\omega' t - \alpha) - 2\,b\,A\omega' \sin (\omega' t - \alpha) + \omega^2 A \cos (\omega' t - \alpha)$

$$= f \cos (\omega' t - \alpha) \cos \alpha - f \sin (\omega' t - \alpha) \sin \alpha$$

As this equation is true for all values of t, the coefficients of $\cos (\omega' t - \alpha)$ and that of $\sin (\omega' t - \alpha)$ on both sides must be separately equal.

Hence, we can write,

$$-A\omega'^2 + \omega^2 A = f \cos \alpha$$

or, $$f \cos \alpha = A\,(\omega^2 - \omega'^2) \quad \text{...(3.3)}$$

and $$-2b\,A\omega' = -f \sin \alpha$$

or, $$f \sin \alpha = 2\,b\,A\omega' \quad \text{...(3.4)}$$

Now, squaring and adding Eqs. (3.3) and (3.4), we get,

$$f^2 = A^2\,(\omega^2 - \omega'^2)^2 + 4\,A^2 b^2 \omega'^2$$

or, $$A = \frac{f}{\sqrt{(\omega^2 - \omega'^2)^2 + 4b^2\,\omega'^2}} \quad \text{...(3.5)}$$

Again dividing Eq. (3.4) by Eq. (3.3) we get,

$$\tan \alpha = \frac{2b\omega'}{\omega^2 - \omega'^2} \quad \text{...(3.6)}$$

Now, considering Eqs. (3.2), (3.5) and (3.6), we get, the solution of Eq. (3.1) in the following form:

$$y = \frac{f}{\sqrt{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}} \cos (\omega' t - \alpha) \quad \text{...(3.7)}$$

where $\alpha = \tan^{-1}\left(\dfrac{2b\omega'}{\omega^2 - \omega'^2}\right)$

The aforeshown solution of Eq. (3.1) given by Eq. (3.7) is for the steady-state condition. But at the starting, as the force sets the particle into vibration, the particle will keep on vibrating with its natural frequency along with the frequency of the forcing system. So the applied sinusoidal force will be acting as damping force and hence the solution for displacement will be given by

$$y = pe^{-bt}\cos\left\{\left(\sqrt{\omega^2 - b^2}\right) t - \theta\right\} \qquad \text{... (3.8)}$$

where p and θ are constants [vide Eq. (2.8) of damping vibration in Chapter 2].

So, the complete solution at the starting is given by,

$$y = pe^{-bt}\cos\left\{\left(\sqrt{\omega^2 - b^2}\right) t - \theta\right\} + \frac{f}{\sqrt{\{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2\}}}\cos\left\{\omega' t - \tan^{-1}\left(\frac{2b\omega'}{\omega^2 - \omega'^2}\right)\right\} \qquad \text{...(3.9)}$$

Thus, as seen above in Eq. (3.9), the solution is made up of two components.

(a) The first component (term) of the solution represents a free vibration with natural frequency set up in a damped system created by the influencing force which has been applied externally. The amplitude is decaying exponentially at a rate which is determined by the damping factor b.

(b) The second component (term) represents a simple harmonic motion of a constant amplitude having the same period $T = \dfrac{2\pi}{\omega'}$ as the applied periodic force and lagging behind the forcing system by a phase angle of α.

Following the lapse of sufficient time, when the steady state is reached, the first damping term of the solution reduces to zero. If the damping factor is very small, then the natural vibration of the system will persist for a longer period of time since e^{-bt} will fall very slowly. It is only the large damping which prevents the first term from persisting indefinitely.

3.4 ENERGY OF FORCED VIBRATIONS: ENERGY RESONANCE

In the steady state under the influence of a periodic force, the motion of a particle is obtained in Eqs. (3.2) as given below:

$$y = A\cos(\omega' t - \alpha) \quad \text{by ...(3.2)}$$

where $A = \left(\dfrac{f}{\sqrt{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}}\right)$

and $\alpha = \tan^{-1}\left(\dfrac{2b\omega'}{\omega^2 - \omega'^2}\right)$

So, $\dfrac{dy}{dt} = -A\omega'\sin(\omega' t - \alpha)$

Hence, the kinetic energy of the forced system at any instant is given by,

$$T = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 = \frac{1}{2}mA^2\omega'^2\sin^2(\omega' t - \alpha)$$

or, $$T = \frac{1}{2}\cdot\frac{m\omega'^2 f^2}{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}\sin^2(\omega' t - \alpha) \qquad \text{...(3.10)}$$

Since, the motion of the particle is steady SHM, the total energy of the system at any instant is equal to the maximum kinetic energy T_{max}.

$$\therefore \qquad E_{total} = T_{max} = \frac{1}{2}\frac{m\omega'^2 f^2}{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}$$

or,
$$E_{total} = \frac{1}{2}\frac{mf^2}{\left(\frac{\omega^2}{\omega'} - \omega'\right)^2 + 4b^2}$$

or,
$$E_{total} = \frac{1}{2}\frac{mf^2}{\omega^2\left(\frac{\omega}{\omega'} - \frac{\omega'}{\omega}\right)^2 + 4b^2}$$

or,
$$E_{total} = \frac{1}{2}\frac{mf^2}{\Delta^2\omega^2 + 4b^2} \qquad ...(3.11)$$

where $\Delta = \frac{\omega}{\omega'} - \frac{\omega'}{\omega}$

If $\omega = \omega'$, then $\Delta = 0$, and the energy of the system is maximum for a given (i.e., particular) value of b. **Thus we observe that when the frequency of the forcing system coincides with that of the natural frequency of the forced system, the energy of vibration of the forced system is maximum. This phenomenon is known as energy resonance or velocity resonance or simply resonance.** And the energy at resonance, E_r, is given by,

$$E_r = E_{max} = \frac{1}{2}\frac{mf^2}{(4b^2)} \qquad ...(3.12)$$

Now, from Eq. (3.11) it is clear that the decrease in the energy due to mistuning between the two frequencies $v = \left(\frac{\omega}{2\pi}\right)$ and $v' = \left(\frac{\omega'}{2\pi}\right)$ is same for a given ratio ($v : v'$) of the frequencies and it is independent of their sign.

3.5 SHARPNESS OF RESONANCE

Since the frequency of the influencing applied periodic force differs from the natural frequency of the forced system, the response of the forced system diminishes. The energy of response of the forced system falls off rapidly for slight deviation from the resonance. The E_{total} versus Δ graph has been plotted in Fig. 3.2. The graph shows that for slight variation of ω' from ω, the total energy gets reduced rapidly.

The sharpness of the resonance curve is a measure of the rate of fall of energy of resonance with departure from equality between the frequencies. From Eqs. (3.11) and (3.12), we obtain,

$$\frac{E_{max}}{E_{total}} = \frac{\Delta^2\omega^2 + 4b^2}{4b^2} = 1 + \frac{\omega^2\Delta^2}{4b^2} \qquad ...(3.13)$$

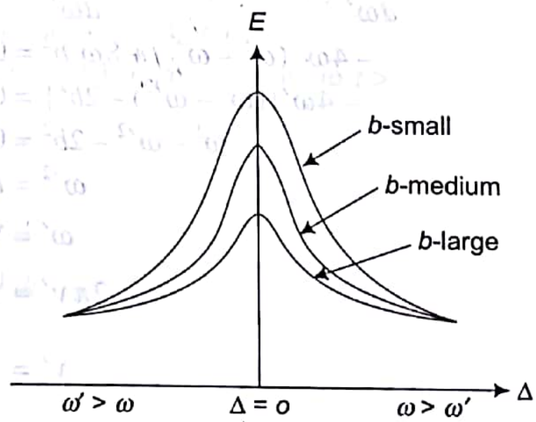


Fig. 3.2 The energy resonance curve sharpness of E versus Δ graph changes rapidly for small change in the damping factor b.

Now, dropping the suffix of E_{total}, we get

$$\frac{E_{max}}{E} = 1 + \frac{\Delta^2\omega^2}{4b^2} \quad \text{... (3.14)}$$

The rate of decrease of total energy E with respect to Δ is greater, the smaller the value of b. The sharpness of resonance can be quantitatively defined as the reciprocal of Δ when E reduces to half of its resonance value.

Let us assume that $\Delta = \Delta_1$ for which $E = \frac{1}{2}E_{max}$. Then from Eq. (3.14) we get,

$$2 = 1 + \frac{\omega^2\Delta_1^2}{4b^2}$$

or, $$4b^2 = \Delta_1^2\omega^2 \Rightarrow \Delta_1 = \pm\frac{2b}{\omega}$$

So, the sharpness of resonance (S) is given by,

$$S = \frac{1}{\Delta_1} = \frac{\omega}{2b} \quad \text{...(3.15)}$$

The sharp resonance is important in connection with a radio receiver.

3.6 AMPLITUDE RESONANCE

In case of forced vibration the amplitude is given by,

$$A = \frac{f}{\sqrt{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}} \Rightarrow A^2 = \frac{f^2}{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2} \quad \text{...(3.16)}$$

A will be maximum when the denominator becomes minimum. For a given system, b and ω are constants. And for the denominator to be minimum, we must have the first derivative of it with respect to ω' to be equal to zero, i.e.,

$$\frac{d}{d\omega'}\{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2\} = 0$$

or, $$\frac{d}{d\omega'}(\omega^2 - \omega'^2)^2 + 4b^2\frac{d\omega'^2}{d\omega'} = 0$$

or, $$-4\omega'(\omega^2 - \omega'^2) + 8\omega' b^2 = 0$$

or, $$-4\omega'\{(\omega^2 - \omega'^2) - 2b^2\} = 0$$

or, $$\omega^2 - \omega'^2 - 2b^2 = 0$$

or, $$\omega'^2 = \omega^2 - 2b^2$$

or, $$\omega' = \sqrt{\omega^2 - 2b^2}$$

or, $$2\pi\nu' = \sqrt{\omega^2 - 2b^2} \quad [\because \omega' = 2\pi\nu'] \quad \text{...(3.17)}$$

or, $$\nu' = \frac{\sqrt{\omega^2 - 2b^2}}{2\pi} \quad \text{...(3.18)}$$

Thus, for the amplitude to be maximum, the frequency must be $\frac{(\sqrt{\omega^2 - 2b^2})}{2\pi}$.

This is called amplitude resonance. This frequency is neither equal to the natural frequency of the system $\left(\nu = \frac{\omega}{2\pi}\right)$ nor equal to the frequency of the damped vibrator $\left[\nu_d = \frac{\sqrt{(\omega^2 - b^2)}}{2\pi}\right]$.

But it is slightly lower than ν_d.

The maximum amplitude at resonance is given by,

$$A_r = A_{\max} = \frac{f}{\sqrt{(\omega^2 - (\omega^2 - 2b^2))^2 + 4b^2(\omega^2 - 2b^2)}} \quad \text{[by Eqs. (3.17) and (3.18)]}$$

or,
$$A_r = \frac{f}{\sqrt{(4b^4 + 4b^2\omega^2 - 8b^4}} = \frac{f}{2b\sqrt{\omega^2 - b^2}}$$

or,
$$A_r = \frac{f}{2b\sqrt{\omega'^2 + b^2}} \quad \text{...(3.19)}$$

$$[\because \omega'^2 = \omega^2 - 2b^2]$$

Hence the maximum amplitude is the greater, the lower is the value of the damping factor b. The sharpness is greater if the damping factor b is smaller.

3.7 THE QUALITY FACTOR

If the driving force $F_p = F \cos \omega' t$ produces an infinitesimal displacement dy in time dt, then the work done or energy supplied by the driving force during the time dt is given by

$$dW = F_p(t)\, dy$$

So, the power supplied is given by

$$p(t) = \frac{dW}{dt} = F_p(t)\frac{dy}{dt}$$

or,
$$p(t) = F \cos \omega' t.\, v$$

or,
$$p(t) = -F \cos \omega' t.\, A\omega' \sin(\omega' t - \alpha)$$

or,
$$p(t) = -FA\omega' \cos \omega' t \sin(\omega' t - \alpha) \quad [\because y = A\cos(\omega' t - \alpha)]$$

$\therefore$
$$P_{av} = \langle p(t)\rangle = \langle -FA\omega' \cos \omega' t \sin(\omega' t - \alpha)\rangle$$

or,
$$P_{av} = -FA\omega' \langle \cos \omega' t(\sin \omega' t \cos \alpha - \cos \omega' t \sin \alpha)\rangle$$

or,
$$P_{av} = -FA\omega' \cos\alpha \langle \cos \omega' t \sin \omega' t\rangle + FA\omega' \sin\alpha \langle \cos^2 \omega' t\rangle$$

But, we know that

$$\langle \cos^2 \omega' t\rangle = \frac{1}{T'}\int_0^{T'} \cos^2 \omega' t\, dt$$

or,
$$\langle \cos^2 \omega' t\rangle = \frac{1}{T'}\int_0^{T'} \cos^2 .\left(\frac{2\pi}{T'}t\right) dt$$

or,
$$\langle \cos^2 (\omega' t)\rangle = \frac{1}{2}$$

where $T' = \frac{2\pi}{\omega'}$ = the time period of the steady-state oscillation.

Similarly,

$$< \cos(\omega' t)\sin(\omega' t) > = \frac{1}{2} < \sin(2\omega' t) >$$

or, $$< \cos(\omega' t)\sin(\omega' t) > = \frac{1}{2T'}\int_0^{T'} \sin\left(\frac{4\pi t}{T'}\right) dt$$

or, $$< \cos(\omega' t)\sin(\omega' t) > = 0$$

Hence, the average power supplied over one cycle is given by

$$P_{av} = \frac{1}{2} FA\omega' \sin\alpha$$

∴ $$P_{av} = \frac{1}{2} F\,\omega' \sin(\alpha)\left(\frac{f}{\sqrt{(\omega^2-\omega'^2)^2+4b^2\omega'^2}}\right).$$ [After the value of A is put]

∴ the maximum value of P_{av} occurs at the frequency ω' by satisfying the following conditions:

$$\frac{d}{d\omega'}\left[\frac{\omega'\sin\alpha}{\sqrt{(\omega^2-\omega'^2)^2+4b^2\omega'^2}}\right] = 0$$

and $$\frac{d^2}{d\omega'^2}\left[\frac{\omega'\sin\alpha}{\sqrt{(\omega^2-\omega'^2)^2+4b^2\omega'^2}}\right] \le 0 \quad \left[\because \tan\alpha = \frac{2b\omega'}{\omega^2-\omega'^2} \therefore \sin\alpha = \frac{2b\omega'}{\sqrt{(\omega^2-\omega'^2)^2+4b^2\omega'^2}}\right]$$

i.e., $$\frac{d}{d\omega'}\left(\frac{2b\omega'^2}{(\omega^2-\omega'^2)^2+4b^2\omega'^2}\right) = 0$$

and $$\frac{d^2}{d\omega'^2}\left(\frac{2b\omega'^2}{(\omega^2-\omega'^2)^2+4b^2\omega'^2}\right) \le 0$$

It is easy to see that these conditions are satisfied for $\omega = \omega'$. In other words, the maximum value of P_{av} (i.e., P_{max}) occurs at resonance.

So, $$P_{max} = \frac{1}{2} F.f.\frac{1}{2b} = \frac{1}{2}\frac{F^2}{m}\frac{1}{2b}$$

or, $$P_{max} = \frac{F^2}{4mb} \quad ...(3.20)$$

∴ $$P_{av} = \frac{1}{2}\times\frac{F^2}{m}\times\frac{2b\omega'^2}{(\omega^2-\omega'^2)^2+4b^2\omega'^2}$$

or, $$P_{av} = \frac{F^2 b}{m}\,\frac{\omega'^2}{(\omega^2-\omega'^2)+4b^2\omega'^2}$$

or, $$P_{av} = \frac{F^2}{4mb}\,4b^2.\frac{\omega'^2}{(\omega^2-\omega'^2)^2+4b^2\omega'^2}$$

or, $$P_{av} = P_{max}\frac{4b^2\omega'^2}{(\omega^2-\omega'^2)^2+4b^2\omega'^2} \quad ...(3.21)$$

Figure 3.3 depicts the variation of P_{av} as a function of ω'.

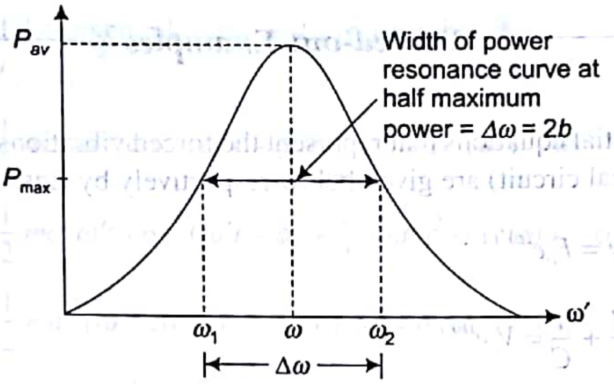


Fig. 3.3 Variation of power P_{av} (at power resonance) as a function of angular frequency (ω') of the driving force.

The values of ω' at which P_{av} is half of its maximum value are called half power points.

From Eq. (3.21) the half power points are the values of ω' satisfying the equation given below:

$$\frac{P_{av}}{P_{max}} = \frac{1}{2} = \frac{4b^2\omega'^2}{(\omega^2 - \omega'^2)^2 + 4b^2\omega^2}$$

or,

$$\omega'^2 = \omega^2 \pm 2b\omega'$$

These are two quadratic equations in ω' namely,

$$\omega'^2 + 2b\omega' - \omega^2 = 0$$

and

$$\omega'^2 - 2b\omega' - \omega^2 = 0$$

Each of these two equations has one positive root and one negative root. Since the angular frequency ω' cannot be negative, we retain only the positive roots, which are

$$\omega_1 = -b + (\omega^2 + b^2)^{\frac{1}{2}}$$

and

$$\omega_2 = +b + (\omega^2 + b^2)^{\frac{1}{2}}$$

The frequency interval between the two half-power points is given by,

$$\Delta\omega = \omega_2 - \omega_1 = 2b$$

The frequency interval $\Delta\omega$ between the two half-power-points is called full frequency with at half-maximum power or simply bandwidth.

The sharpness of resonance can also be measured in terms of quality factor, or Q-factor, which is defined as follows:

$$Q = \frac{\omega}{\omega_2 - \omega_1} = \frac{\omega}{\Delta\omega} = \text{resonant frequency/bandwidth}$$

or,

$$Q = \frac{\omega}{2b} = S$$

where S is the sharpness.

Worked-out Examples

Example 3.1 The differential equations that represent the forced vibrations of one mechanical system and one electrical system (electrical circuit) are given below respectively by Eqs. (1) and (2):

$$m\frac{d^2y}{dt^2} + k'\frac{dy}{dt} + ky = F_o e^{i\omega' t} \quad \text{...(1)}$$

and
$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_o e^{i\omega' t} \quad \text{...(2)}$$

where ω' is the angular frequency of the applied force, $j = \sqrt{-1}$ and k' is resistive force per unit velocity. Make a table of two columns by listing the analogous parameters of the two systems.

Sol. Table of analogues

Mechanical quantities	*Electrical quantities*
Force (F)	Voltage (V)
Displacement (y)	Charge (q)
Velocity (v)	Current (i)
Mass (m)	Inductance (L)
Mechanical resistance (k')	Resistance (R)
Compliance ($1/k$)	Capacitance (C)
Mechanical impedance (z_m)	Impedance (Z)
Mechanical reactance (X_m)	Reactance (X)
Internal reactance ($\omega' m$)	Inductive reactance ($\omega' L$)
Reactance of compliance (k'/ω')	Capacitive reactance ($1/\omega' C$)

Example 3.2 Show that for a system of forced vibration, the total energy of vibrating system is not constant and that, in such a case,

$$\frac{\text{average potential energy}}{\text{average kinetic energy}} = \frac{\omega^2}{\omega'^2}$$

where ω = natural angular frequency = $\sqrt{\frac{k}{m}}$ and ω' is the angular frequency of the external periodic force.

Sol. The displacement y of the particle of mass m in forced vibration, is given by

$$y = a\sin(\omega' t - \phi)$$

and velocity
$$v = \dot{y} = a\omega'\cos(\omega' t - \phi)$$

$\therefore$ the kinetic energy of the particle E_k is given by,

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}m\,a^2\omega'^2\cos^2(\omega' t - \phi)$$

and the potential energy of the particle E_P is given by,

$$E_P = \frac{1}{2}ky^2 = \frac{1}{2}k\,a^2\sin^2(\omega' t - \phi)$$

Sum of the kinetic and the potential energies E_S is given by

$$E_s = E_k + E_P$$

or, $$E_s = \frac{1}{2} ma^2\omega'^2 \cos^2(\omega' t - \phi) + \frac{1}{2} ka^2 \sin^2(\omega' t - \phi)$$

or, $$E_s = \frac{1}{2} ma^2\omega'^2 \cos^2(\omega' t - \phi) + \frac{1}{2} m\omega^2 a^2 \sin^2(\omega' t - \phi) \quad \left[\because \omega = \sqrt{\frac{k}{m}}\right]$$

or, $$E_s = \frac{1}{2} ma^2 [\omega'^2 \cos^2(\omega' t - \phi) + \omega^2 \sin^2(\omega' t - \phi)]$$

Hence, $$E_s \neq \text{constant} \quad [\because \omega \neq \omega']$$

Now, $$\overline{E}_k = (E_k)_{av} = \frac{1}{2} ma^2\omega'^2 \frac{1}{2}$$

$$\left[\because \text{over a full cycle, the average value of } \cos^2(\omega' t - \phi) \text{ is } \frac{1}{2}\right]$$

So, $$\overline{E}_k = \frac{1}{4} ma^2\omega'^2 \qquad \text{... (1)}$$

Similarly, $$\overline{E}_p = \frac{1}{4} ma^2\omega^2 \qquad \text{...(2)}$$

$$\left[\because \text{over a full cycle, the average value of } \sin^2(\omega' t - \phi) \text{ is } \frac{1}{2}\right]$$

Hence, the ratio of $\overline{E}_P$ to $\overline{E}_K$ (demoted by R_{pk}) is given by

$$R_{pk} = \frac{\overline{E}_P}{\overline{E}_k} = \frac{\frac{1}{4} ma^2\omega^2}{\frac{1}{4} ma^2\omega'^2} = \frac{\omega^2}{\omega'^2}$$

Hence proved.

Example 3.3 Find an expression for the fractional change in natural frequency of a damped (simple) harmonic oscillator in terms of the quality factor Q.

Sol. Let ω' and ω be the angular frequencies of the damped and the undamped oscillator respectively. Then, we have,

$$\omega' = \sqrt{\omega^2 - b^2}$$

where $$b = \frac{k'}{2m}$$

and k' = resistive force per unit velocity

or, $$\omega' = \sqrt{\omega^2 - \left(\frac{k'}{2m}\right)^2}$$

or, $$\omega' = \omega\left(1 - \frac{k'^2}{4m^2\omega^2}\right)^{\frac{1}{2}}$$

Now, if k' is small then $\dfrac{k'^2}{4m^2\omega^2}$ is also very small.

So, by using binomial expression and neglecting higher terms we can write,

$$\omega' = \omega\left(1 - \frac{k'^2}{8\,m^2\omega^2}\right)$$

$$\therefore \quad \frac{\omega'}{\omega} = 1 - \frac{k'^2}{8\,m^2\omega^2}$$

or, $$1 - \frac{\omega'}{\omega} = \frac{k'^2}{8\,m^2\omega^2}$$

or, $$\frac{\omega - \omega'}{\omega} = \frac{k'^2}{8\,m^2\omega^2}$$

or, $$\frac{\Delta\omega}{\omega} = \frac{k'^2}{8\,m^2\omega^2}$$

or, $$\frac{\Delta\omega}{\omega} = \frac{1}{8\omega^2}\left(\frac{k'}{m}\right)^2 = \frac{1}{8\omega^2}\,(2b)^2 \quad \left[\because \frac{k'}{m} = 2b\right]$$

or, $$\frac{\Delta\omega}{\omega} = \frac{1}{8}\cdot\left(\frac{2b}{\omega}\right)^2 = \frac{1}{8}\cdot\frac{1}{\left(\frac{\omega}{2b}\right)^2}$$

or, $$\frac{\Delta\omega}{\omega} = \frac{1}{8Q^2} \quad \left[\because Q = \frac{\omega}{2b}\right]$$

$\therefore$ fractional change in natural frequency (f_c) is given by

$$f_c = \frac{1}{8Q^2}$$

Example 3.4 A voltage having root mean square value V_{rms} = 100 volts is applied to a series resonant circuit with resistance R = 10 ohms, inductance L = 10 mH and capacitance C = 1μF. Calculate the natural frequency and current at resonance in the circuit.

Sol. From the table of analogues of Example 3.1, we can write the following table of analogues:

Mechanical quality	*Electrical quality*
Mass (m)	Inductance (L)
Compliance ($1/k$)	Capacitance (C)
Mechanical resistance (k')	Resistance (R)

In case of a mechanical system, the natural frequency is given by

$$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\left(\frac{1}{m}\right)\left(\frac{1}{1/k}\right)}$$

$\therefore$ in case of the electrical circuit the natural frequency is given by

$$\nu = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

or, $$\nu = \frac{1}{2 \times 3.14 \times \sqrt{10 \times 10^{-3} \times 1 \times 10^{-6}}}$$

or, $$\nu = 1592 \text{ Hz}$$

The current at resonance is given by

$$I_{max} = \frac{V_{max}}{R} = \frac{\sqrt{2}\, V_{rms}}{R} \quad [\because V_{max} = \sqrt{2}\, V_{rms}]$$

or, $$I_{max} = \frac{\sqrt{2} \times 100}{10} = 14.14 \text{ A}$$

Example 3.5 An object of mass m = 1 g hangs from a spring of spring constant $k = 10^6$ dyne cm^{-1}. A resistive force $k'v$ acts on it where v is the velocity in cm s^{-1} and $k' = 10^4$ g s^{-1}. If the object be subjected to a driving force, $F = F_p \cos \omega' t$ with $F_p = 2 \times 10^6$ dyne and $\omega' = 5000$ rad s^{-1}, then calculate the amplitude of oscillation and phase, relative to the applied resistive force, in the steady state.

Sol. The spring constant $k = 10^4$ = dyne cm^{-1}, mass m = 1g

∴ natural angular frequency of the object is given by

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{10^6}{1.0}} = 1000 \text{ rad s}^{-1}$$

∴ the force acting per unit mass is given by

$$f_p = \frac{F_p}{m} = \frac{2 \times 10^6}{1} = 2 \times 10^6 \text{ dyne g}^{-1}$$

and $$b = \frac{k'}{2m} = \frac{10^4}{2 \times 1} = 5 \times 10^3 \text{ s}^{-1}$$

∴ in the steady state, the amplitude of oscillation is given by,

$$A = \frac{f_p}{\sqrt{4b^2\omega'^2 + (\omega'^2 - \omega^2)^2}} = \frac{2 \times 10^6}{\sqrt{4 \times 25 \times 10^6 \times 25 \times 10^6 + (25 \times 10^6 - 10^6)^2}}$$

or, $$A = \frac{2 \times 10^6}{\sqrt{25 \times 10^{14} + 576 \times 10^{12}}} = \frac{2 \times 10^6}{10^7 \times \sqrt{25 + 5.76}}$$

or, $$A = \frac{2 \times 10^{-1}}{\sqrt{30.76}} = 3.606 \times 10^{-2} \text{ cm}$$

If α be the phase by which the oscillations of the driven system lag behind the force then

$$\tan \alpha = \frac{2b\omega'}{\omega'^2 - \omega^2} = \frac{2 \times 5 \times 10^3 \times 5 \times 10^3}{25 \times 10^6 - 1.0 \times 10^6}$$

or, $$\tan \alpha = \frac{50 \times 10^6}{24 \times 10^6} = \frac{25}{12} = \tan^{-1} 64.35°$$

∴ $$\alpha = 64.35°$$

Example 3.6 Compare the phase of the driven system with respect to that of the driving force.

Sol. From Eq. (3.6),we have,

$$\tan \alpha = \frac{2b\omega'}{(\omega^2 - \omega'^2)} \quad ...(1)$$

Let us now observe the variation of the phase α of the driven system (i.e., phase lag) with respect to the angular frequency ω' of the applied external force. Let us vary ω' from zero to infinity, in a few steps, and observe the impact on the phase α.

(i) If we put $\omega' = 0$ in the equation, we get, $\tan \alpha = 0$ which implies $\alpha = 0$.
Thus, there is no plase difference between the driving force and the driven system.

(ii) When $\omega' < \omega$, from Eq. (1), we get, $\tan \alpha > 0$.
Thus, the phase difference α has a value which lies between 0 and $\frac{\pi}{2}$.

(iii) If we put $\omega' = \omega$ in Eq. (1), we get, $\tan \alpha = \infty$. This implies $\alpha = \frac{\pi}{2}$
Thus, at resonance, the driven system lags behind the driving force by an angle of $\frac{\pi}{2}$

(iv) When $\omega' > \omega$, we can observe from Eq. (1), we get, $\tan \alpha < 0$.
Thus, the phase difference α has a value which lies between $\frac{\pi}{2}$ and π and α lies in the second quadrant.

(v) When $\omega' \rightarrow, \infty$ we can observe from Eq. (1) that $\tan \alpha \rightarrow 0$.
Thus when $\omega' = \infty$, $\alpha = \pi$

Now, from the above discussion, we find that for all values of ω', the phase difference α lies between 0 and π and it becomes equal to $\frac{\pi}{2}$ at resonance, i.e., when $\omega' = \omega$, $\alpha = \frac{\pi}{2}$.

Example 3.7 Show that at the resonance frequency, the rate of change of phase angle (with respect to natural frequency of the driven system) is inversely proportional to b where $b = \frac{k'}{2\,m}$ and k' is damping force per unit velocity of the driven system. And m is the mass of the system.

Sol. From Equation (3.6), we get,

$$\tan \alpha = \frac{2b\omega'}{(\omega^2 - \omega'^2)} \quad ...(1)$$

or,
$$\alpha = \tan^{-1}\left(\frac{2b\omega'}{\omega^2 - \omega'^2}\right)$$

or,
$$\frac{d\alpha}{d\omega} = \frac{-4b\omega'\,\omega}{4b^2\omega'^2 + (\omega^2 - \omega'^2)^2} \quad ...(2)$$

When $\omega' = \omega$, resonance takes place.

$\therefore$ at resonance,
$$\frac{d\alpha}{d\omega} = -\frac{1}{b}$$

or,
$$\frac{d\alpha}{d\,(2\pi\nu)} = -\frac{1}{b} \text{ where } \omega = 2\pi\nu$$

or, $$\frac{d\alpha}{d\nu} = -\frac{2\pi}{b}$$

$\therefore$ $$\frac{d\alpha}{d\nu} \propto \frac{1}{b}$$

Hence, the rate of change of phase angle with respect to frequency (ν) of the driven system is inversely proportional to b.

Review Exercises

Part 1: Multiple Choice Questions

1. Which is the case of a forced vibration?
   (a) Vibrations produced in the string of a piano
   (b) Vibrations produced in a telephone transmitter during conversation
   (c) Sound produced in an organ pipe
   (d) None of these
2. Two vibrating systems are said to be in resonance if
   (a) their amplitudes are equal (b) their frequencies are equal
   (c) they are in same phase (d) none of these
3. The resonant frequency of an electrical oscillator is given by
   (a) $\nu = 2\pi\sqrt{LC}$ (b) $\nu = \frac{1}{2\pi\sqrt{LC}}$
   (c) $\nu = \frac{2\pi}{\sqrt{LC}}$ (d) none of these
4. A damped harmonic oscillator of frequency ω is acted upon by an external force $F = F_o \sin \omega t$. If the natural frequency of free oscillations be ω_o, the frequency of oscillations of the forced oscillations is in steady state will be
   (a) ω_o (b) $\omega - \omega_o$ (c) ω (d) $\frac{\omega + \omega_o}{2}$
5. For small value of damping constant, the resonance is
   (a) flat (b) sharp (c) remains same (d) none of these
6. In a forced oscillator at very low and high frequencies the value of which of the following variables tends to zero.
   (a) Phase (b) Charge (c) Driving emf (d) None of these
7. What is the phase difference between the driving force and the velocity of the forced oscillator?
   (a) ϕ (b) $\frac{\pi}{2} + \phi$ (c) $-\phi + \frac{\pi}{2}$ (d) $\phi + \pi$
8. For a large value of the damping constant, the resonance is
   (a) sharp (b) flat (c) remains same (d) none of these

9. Which of the following is the phase (ϕ) relationship between the displacement y of the forced oscillator and the applied force F?
   (a) y lags behind F by ϕ (b) y leads F by ϕ
   (c) y lags behind F by $\pi/2$ (d) y leads F by $\pi/2$
10. Considering the equation of the forced vibrations as $m\frac{d^2y}{dt^2} + k'\frac{dy}{dt} + ky = F_o \sin \omega' t$ which one of the following statements is correct?
    (a) k' is a resistive force per unit velocity
    (b) k' is the stiffness factor
    (c) k' is the angular frequency of the applied external force
    (d) None of these
11. The maximum amplitude in forced vibration is given by
    (a) $\frac{f}{\sqrt{\omega'^2 + b^2}}$ (b) $\frac{f}{2b\sqrt{\omega'^2 + b^2}}$
    (c) $\frac{f}{2b\sqrt{\omega'^2 - b^2}}$ (d) $\frac{f}{2b(\omega' - b)}$
12. When the frequency (ω') of the applied force and the natural frequency of vibration (ω) are equal, the value of maximum amplitude is given by
    (a) $\frac{f}{\sqrt{2b\omega'}}$ (b) $\frac{f}{2b\omega'}$ (c) $\frac{f}{2b}$ (d) none of these
13. In amplitude resonance, the frequency of the applied external force is given by
    (a) $v' = \frac{\sqrt{\omega^2 - b^2}}{2\pi}$ (b) $v = \frac{\sqrt{b^2 - 2\omega^2}}{2\pi}$
    (c) $v' = \frac{\sqrt{\omega^2 - 2b^2}}{2\pi}$ (d) none of these
14. In amplitude resonance, the relation between natural frequency of forced system (ω) and the frequency of the applied force (ω') is given by
    (a) $\omega' = \sqrt{\omega^2 + 2b^2}$ (b) $\omega' = \sqrt{\omega^2 - 2b^2}$
    (c) $\omega' = \sqrt{\omega^2 - b^2}$ (d) none of these
15. The relation between the sharpness (S) and the quality factor (Q) in case of forced vibration is given by
    (a) $Q = 2S$ (b) $Q = \frac{1}{S}$ (c) $Q = \frac{1}{2S}$ (d) $Q = S$

[**Ans.** 1(b), 2(c), 3(b), 4(c), 5(b), 6(d), 7(a), 8(b), 9(a), 10(a), 11(b), 12(b), 13(c), 14(b), 15(d)]

Short Questions with Answers

1. **What do you mean by forced vibration?**

Ans. By forced vibration we mean a phenomenon in which a body is set into vibration with the help of a strong external periodic force having a frequency different from the natural frequency of the body itself.

2. Why is a large amplitude produced when frequency of the applied external periodic force is equal to the natural frequency of the forced body?

Ans. If the frequency of the external periodic force applied is equal to the natural frequency of the body, then the external applied force increases the amplitude of vibration of the body as the external applied force is always in phase with the velocity of vibration of the body. Each new impulse adds to the effect of all the previous impulses because the successive impulses always arrive in phase and the resultant effect makes the amplitude of vibration large.

3. What is meant by velocity resonance?

Ans. In the case of forced vibration, at a certain frequency of the external driving force, the amplitude of velocity of the forced vibrator becomes maximum. By velocity resonance we mean this phenomenon.

4. What is meant by amplitude resonance?

Ans. In case of forced vibration, at a certain frequency of the driving force, the amplitude of the forced vibrator becomes maximum. This is meant by amplitude resonance.

5. Is the energy supplied by the driving force stored in the forced vibrator? Explain.

Ans. No, the energy supplied by the driving force is not stored in the forced vibrator. The energy in each cycle of the vibrator gets lost due to damping offered by the medium in which it vibrates. The energy which is supplied by the external periodic applied force is exactly equal to the aforesaid loss of the energy. The externally supplied energy is used to maintain the vibrations of the forced vibrator. And in the steady state, the amplitude and period of vibration of the vibrator are so adjusted that the power supplied by the external periodic applied force is exactly equal to the energy dissipated against the frictional forces offered by the medium.

6. Why are soldiers usually ordered to break their steps when they march over a suspension bridge?

Ans. While marching over a suspension bridge, the soldiers are prohibited to step in unison, because in such a situation, the bridge will vibrate violently due to resonance and it will collapse. The resonance will take place when the frequency of marching of the soldiers will be equal to the natural frequency of the bridge.

7. State the physical significance of the quality factor (Q) of a forced vibrator.

Ans. The quality factor (Q) gives the measure of sharpness of the resonance. The higher the quality factor (Q), the smaller is the bandwidh of resonance and accordingly the tuning will be sharper. This fact is used to increase the selectivity of the radio sets. The sharpness of response of a circuit allows one to reproduce the radio signals without any interference from the signals of frequencies which are close to it. It also accounts for the amplification factor of the signal. And at displacement resonance, the displacement at low frequencies is amplified by a factor of Q.

8. What is the difference between amplitude resonance and velocity resonance?

Ans. Both the amplitude resonance and velocity (or energy) resonance gives the indication that the driven system is vibrating at its natural frequency and the frequency of the driver system. But in case of amplitude resonance, one considers the maximum value of the amplitude while in the other case, one considers the maximum value of the velocity of the vibrator (i.e., the driver system).

9. What is the Q-factor (quality factor) of a forced oscillator? **[WBUT 2008]**

Ans. It is such a factor which can give the exact idea regarding the sharpness of the resonance peak. It is defined as follows:

$$Q = \frac{\omega}{\omega_2 - \omega_1} = \frac{\omega}{\Delta\omega} = \frac{\omega}{2b}$$

where ω is the natural angular frequency of the driven system. And ω_1 and ω_2 are the two values of the angular frequency at which the power dissipation becomes half of its maximum value. $\omega_1 \sim \omega_2$ is known as bandwidth. In short, the quality factor gives an idea regarding the quality or sharpness of the resonance peak.

10. Show that at velocity resonance the velocity is in phase with the driving force [WBUT 2008]

Ans. In the steady state under the influence of a periodic applied force, the displacement of the vibrating system is given by

$$y = A\cos(\omega' t - \alpha)$$

where $$A = \frac{f}{\sqrt{(\omega^2 - \omega'^2) + 4b^2\omega'^2}} \text{ and } \alpha = \tan^{-1}\left(\frac{2b\omega}{\omega^2 - \omega'^2}\right).$$

And α being the phase difference between the applied force F and the displacement y, ω and ω' are respectively the angular frequency of the vibrating system and that of the applied force.

The velocity of the vibrating system is given by

$$v = \frac{dy}{dt} = -A\omega' \sin(\omega' t - \alpha)$$

or, $$v = A\omega' \cos\left(\frac{\pi}{2} + \omega' t - \alpha\right)$$

or, $$v = A\omega' \cos(\omega' t + \phi)$$

where $\phi = \frac{\pi}{2} - \alpha$. ϕ is the phase difference between the applied force and the velocity of the vibrator.

At velocity resonance, the velocity of the vibrating system becomes maximum.

$\therefore$ $$v_{max} = \left(\frac{dy}{dt}\right)_{max} = [A\omega' \cos(\omega' t + \phi)]_{max}$$

or, $$v_{max} = [A\omega']_{max} \times [\cos(\omega' t + \phi)]_{max}$$

or, $$v_{max} = (A\omega']_{max}$$

or, $$v_{max} = \left[\frac{f\omega'}{\sqrt{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}}\right]_{max}$$

or, $$v_{max} = \frac{f\omega'}{2b\omega'} = \frac{f}{2b}$$

Thus, we get the maximum value of v (i.e., v_{max}) only when $\omega' = \omega$.

So, at velocity resonance, the phase difference between the applied periodic force and the velocity of the vibrating system is given by

$$\phi = \frac{\pi}{2} - \alpha = \frac{\pi}{2} - \tan^{-1}\left(\frac{2b\omega'}{\omega^2 - \omega'^2}\right)$$

or, $$\phi = \frac{\pi}{2} - \tan^{-1}(\infty) \quad [\because \omega = \omega']$$

or, $\phi = \frac{\pi}{2} - \frac{\pi}{2} = 0$

Hence, at velocity resonance, the velocity of the vibrating system and applied periodic force are in phase.

Part 2: Descriptive Questions

1. What is the difference between forced oscillation and resonance? Explain what is meant by sharpness of resonance.
2. What is resonance? Draw the graphs of displacement amplitude and velocity amplitude against the frequency of the sinusoidal force driving a mechanical oscillator for different values of damping.
3. Derive an expression for the velocity of a forced oscillator. Discuss the variation of velocity amplitude with driving force frequency and show its behaviour graphically.
4. Distinguish between amplitude resonance and velocity resonance. Show that at velocity resonance, the maximum velocity is inversely proportional to damping and the velocity of the oscillator is in phase with the periodic external force in the steady state.
5. Discuss the behaviour of displacement frequency versus driving force frequency in case of a forced oscillator. Prove that the resultant frequency of the driving force is slightly less than the natural frequency of the displacement.
6. What is the difference between natural vibration and forced vibration? An oscillator can have more than one frequencies of forced vibration but it can have only one frequency of natural vibration. Explain.
7. Distinguish between free and forced vibrations. Write down the equation of forced vibration and solve it. Explain what is meant by Q-factor.
8. Establish the condition for amplitude resonance and explain the sharpness of amplitude resonance.
9. Draw the variation of velocity amplitude against the applied frequency ω' for different damping constants. Prove that the maximum power intake by the oscillator is observed at velocity resonance.
10. Define sharpness of resonance. Find the energy dissipated per unit time per unit mass for forced vibration.

Part 3: Numerical Problems

1. A body of 10 g mass is acted upon by a restoring force per unit displacement of 10^7 dyne/(cm s^{-1}) and a driving force per unit velocity of 4×10^3 dyne/(cm s^{-1}) and a driving force of $10^5 \cos(\omega t)$ dyne. Find the value of maximum amplitude.
2. A particle of 0.01 kg mass is subjected to a restoring force of 0.5 Nm^{-1}, a damping force of 10^{-4} kg s^{-1} and an external sinusoidal force of constant amplitude. Find the frequency of the driving force for which there will be velocity resonance.
3. The forced harmonic oscillations have displacement amplitudes at frequencies $\omega_1 = 400$ rad s^{-1} and $\omega_2 = 600$ rad s^{-1}. Find the resonant frequency at which the displacement amplitude is maximum.
4. An object of 0.1 kg mass hangs from a spring of spring constant of 100 Nm^{-1}. A resistive force rv acts on it where v is the velocity in m s^{-1} and $r = 1$ kg s^{-1}. If the object be subjected to a driving force $F = F_o \cos \omega' t$ with $F_o = 2$ N and $\omega' = 50$ rad s^{-1}, then calculate the amplitude of oscillation and phase, relative to the force, in the steady state.

5. To the lower end of a massless spring of spring-constant of 200 Nm^{-1} and hanging vertically from a rigid support, is connected a particle of 0.1 kg mass. A resistive force $-kv$ acts on the oscillating mass where v is the particle velocity and k is a constant. Find (a) the value of k, (b) The Q-value of the oscillator, and (c) the percentage change in frequency of oscillation due to damping.
6. A particle of 2 g mass is subjected to an elastic force of 0.03 Nm^{-1} and frictional force of 0.0005 N/(cm s^{-1}). If the particle is displaced through 2 cm and then released find whether the resulting motion is oscillatory or not. If so, find its time period.

CHAPTER

4

Interference of Light

4.1 INTRODUCTION

Light is a form of energy and it is regarded as the transfer of energy from luminous sources to the eye, either directly or through the object seen. Energy can be transferred from one point to the other either by means of wave disturbance travelling through the intervening medium or by the motion of material particles between the two points. Accordingly, two different theories about the nature of light wave were brought forward in the seventeenth century; these theories were the corpuscular theory and wave theory. The corpuscular theory is perhaps the simplest theory of light which assumes light to be tiny particles of matter. The corpuscular theory of light was first given by Newton in 1642. According to him, a light source emits tiny corpuscles of light and these corpuscles travel in straight lines in a transparent medium when not acted upon by external forces. It explaining the reflection and refraction of light, the theory says that when light corpuscles approach the exposed surface of material medium, they feel a force of repulsion and attraction. In the case of refraction, the corpuscles are attracted by the refracting surface and enter into the second medium and in the case of reflection, the corpuscles are repelled by the reflecting surface into the first medium. We know that when a beam of light is incident on the surface of a medium, the beam is partly reflected and partly refracted simultaneously. To explain this phenomenon according to the corpuscular theory, it was assumed that some corpuscles are under the action of repulsive force favourable to reflection, others in the same condition are under the action of attractive force favourable to refraction. The arguments put forward by Newton in solving the problem were not very convincing. Further, a large number of experimental observations (like interference, diffraction and polarization) could not be explained on the basis of the corpuscular theory of light.

A Dutch physicist, Christian Huygens (1629–1695), who was a contemporary of Newton, suggested that a satisfactory explanation of interference and diffraction phenomena can only be given if one assumes light as a wave. Using the wave theory, Huygens explained the laws of reflection, refraction and the phenomenon of double refraction. However, no one really believed in Huygens'wave theory until 1801 when Thomas Young (1773–1829) performed the famous interference experiment which only could have explained the phenomenon of interference satisfactorily on the basis of wave theory of light. The corpuscular theory was found to be totally inadequate to explain the experimental results.

Interference is the most general phenomenon in wave motion. Many kinds of waves exhibit interference: such as light waves, sound waves, water waves and so on. The region in which two or more waves are incident on each other is known as the *region of superposition*. The principle of optical interference is based on linear

superposition of electromagnetic waves. In this case, there is no medium like in the case of sound waves. **The modification of intensity due to superposition of two or more beams of light is known as interference of light**. In this chapter, we shall consider the interference pattern produced by two waves emanating from two sources. It may be mentioned here that with sound waves the interference pattern can be obtained easily because the two superposing waves maintain a constant phase relationship. However for light waves, due to the verv process of emission, interference is not observed between the waves from two independent sources. For permanent interference pattern, the two light sources must have constant phase relationship.

The method of achieving these sources and their applications will be discussed below.

4.2 WAVE PROPAGATION AND THE WAVE EQUATION

A wave is the propagation of a disturbance. For example, when a stone is dropped in a calm pool of water at the point P [Fig. 4.1], waves start moving from the point. In this propagation of wave, water goes up and then goes down from the initial level of water. In Fig. 4.1, XY represents the level of water when it is not disturbed. At A and B a portion of water goes up from XY. These are called *crests* whereas at C and D a portion of water goes down from XY. These are called *troughs.* The distance between two crests or two consecutive troughs is called *wavelength* (λ). The distance of crest or trough from the initial horizontal water level is called *amplitude* of the wave and it is denoted by a. When a wave is propagating through the medium the displacement of the water molecule is at right angles to the direction of propagation. This type of wave is known as transverse wave. When a transverse wave propagates, particles of water execute simple harmonic motion (S.H.M.) about their equilibrium position and associated with this motion is a certain amount of energy. **The state of vibration of the vibrating particle is called *phase of the wave*.**

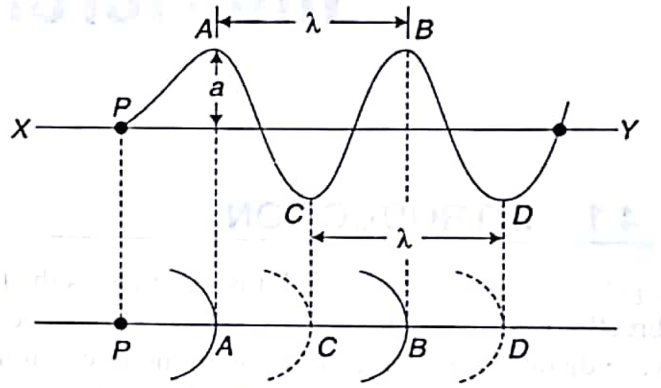


Fig. 4.1 Wave propagation with crests and troughs.

Wave Equation The equation of motion of the particle in SHM is given by

$$y = a \sin \omega t \quad \left[\begin{array}{l}\text{where} \quad a = \text{amplitude} \\ \omega = \text{angular and frequency}\end{array}\right]$$

At a distance x from the first particle, another particle also execute SHM. The phase difference between these two particles having path differences x is given by $\phi = \dfrac{2\pi}{\lambda}x.$

[Since phase difference is 2π when path difference is λ]

So, the equation of motion of the second particle is

$$y = a \sin\left(\omega t - \frac{2\pi}{\lambda}x\right) = a \sin\left(\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x\right)$$

$$= a \sin \frac{2\pi}{\lambda}(vt - x) \left[\text{where phase vel. } v = \frac{\lambda}{T}\right]$$

This is the general equation of wave motion.

4.3 PRINCIPLE OF SUPERPOSITION OF WAVES

The explanation of the phenomenon of interference of light is based on the principle of superposition of wave motion. The principle states that *when two or more waves simultaneously pass through a point of a medium, the resultant displacement at any instant at the point concerned is the algebraic sum of the displacements produced by the individual waves provided the displacements are small.* For example, if two similar waves are oscillating in phase then the net wave amplitude is just double (Fig. 4.2a) and the waves *interfere constructively.* On the other hand, if two waves are exactly 180° out of phase, the net wave amplitude is zero (Fig. 4.2b) and the waves *interfere destructively.*

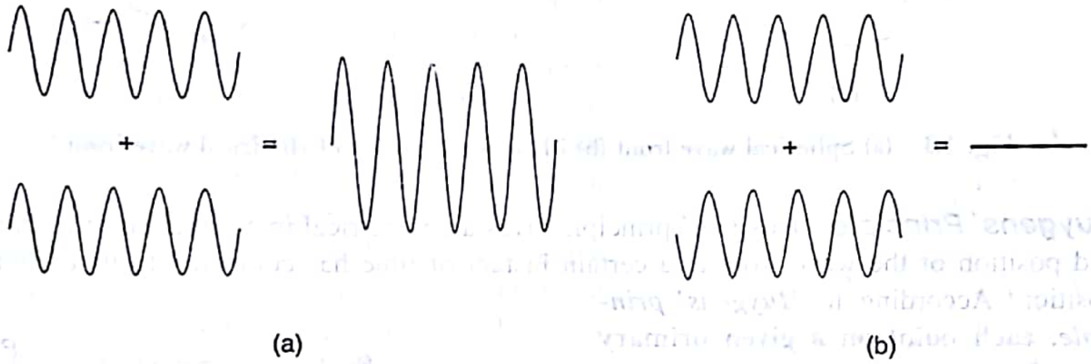


Fig. 4.2 Superposition of waves: (a) the two waves are in phase, (b) the two waves are 180° out of phase.

Note Light is an electromagnetic wave of transverse nature consisting of one electric and one magnetic field vector. The electric field vector characterizes the nature of light. In most of the cases one deals with the situations in which the electric field vector vibrates in a fixed plane. This fixed plane is considered as the plane of vibration and the light is considered to be plane polarized. While analysing interference and diffraction, it is usually considered as the interaction of coplanar waves. In the discussion of interference and diffraction we will also assume light to be plane polarized.

4.4 HUYGENS' WAVE THEORY

(a) Wave front According to the wave theory, a source of light sends out disturbances in all directions. In a homogeneous medium, the points that are situated at equal distance from the source of light will vibrate in the same phase. The locus of all particles vibrating in the same phase at any time is called the ***wave front* of the wave** produced by the vibration of the particles.

For example, if we drop a small stone in a calm pool of water, circular ripples spread out from the point of impact, each point of the circumference of the circle oscillates with the same amplitude and same phase and thus we get a circular wave front.

Depending upon the shape of the source of light, wave front are classified into three categories:

(i) Spherical wave front If we have a point source producing waves in an isotropic homogeneous medium, the locus of points which are in phase is a sphere. In this case, we have spherical wave fronts as shown in Fig. 4.3(a).

(ii) Plane wave front At a large distance from the source, a small portion of the sphere can be considered as a plane and a plane wave front is shown in Fig. 4.3(b).

(iii) ***Cylindrical wave front*** When the source of light is linear in shape, a cylindrical wave front is produced. The curved surface of the cylinder indicates the position of the wave front at a particular instant [Fig. 4.3(c)].

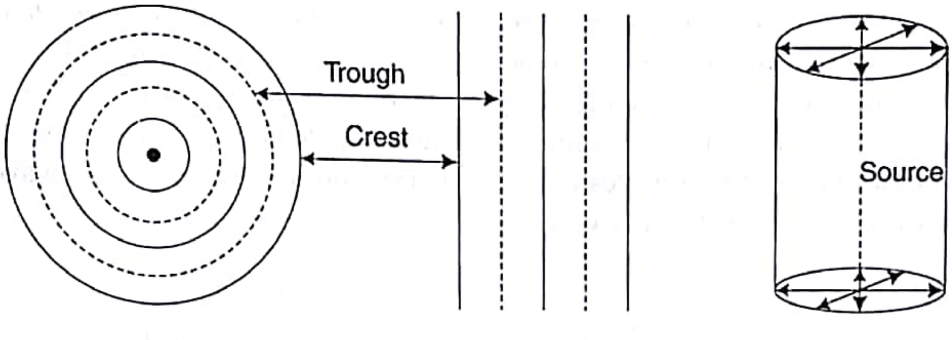


Fig. 4.3 (a) Spherical wave front (b) Plane wave front (c) Cylindrical wave front.

(b) Huygens' Principle Huygens' principle gives a geometrical interpretation of finding the shape and position of the wave front at a certain instant of time has compared to its earlier shape and position. According to ***Huygens' principle*, each point on a given primary wave front acts as a source of secondary wavelets, sending out disturbances in all directions in a similar manner as the original source does. The position of the wave front at the later time (called secondary wave front) is the envelope of the secondary wavelets at that instant.**

In Fig. 4.4(a) PQ represents the shape of the wavefront (originating from the points source S) at $t = 0$. The medium is assumed to be homogeneous and isotropic. Let us now determine the shape of the wavefront after a time interval of Δt. With each point on the wave front as center, let us draw spheres of radius $u\,\Delta t$, where u is the speed of the wave in that medium. The common tangent to all these spheres gives the forward envelope which is again a sphere centered at S. Thus the shape of the new wave front after a time Δt is the sphere $P'Q'$. At a large distance from the source, the radius of wave front will be very large and a part of the wavefront will appear as plane [Fig. 4.4(b)]. We also obtain a backward envelope $P''Q''$. In Huygens' theory, the presence of the backward is avoided by assuming that the amplitude of the secondary wavelets is not uniform in

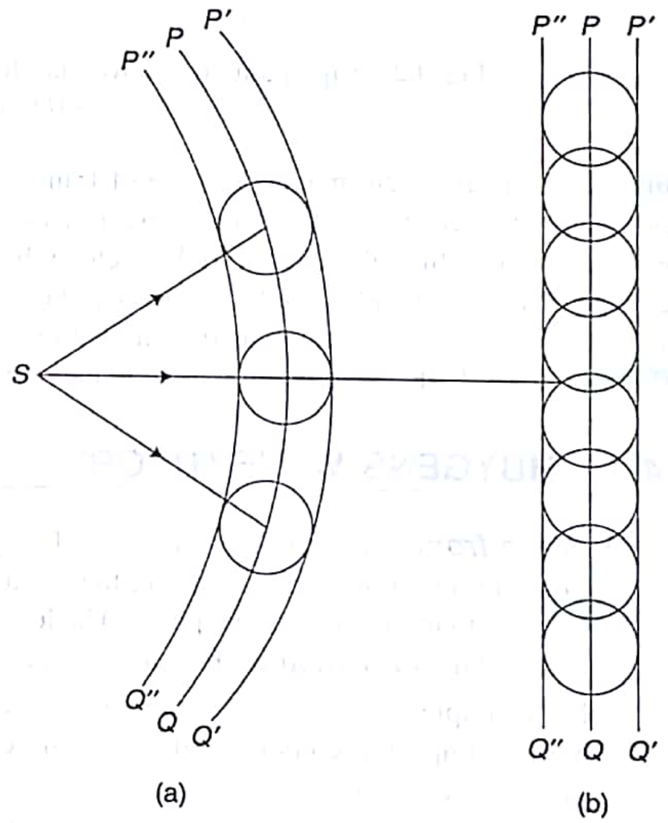


Fig. 4.4 Shape of the wave front (a) PQ is a spherical wave front centered at S at a time say, $t = 0$. After Δt time $P'Q'$ is the forward envelope and $P''Q''$ is the backward envelope. (b) After a large interval of time the distance of the wave from the source S will be very large and consequently the wave front will be a plane wave front.

all directions; it is maximum in the forward direction and zero in the backward direction. The absence of the backward wave is really explained by Fresnel and Kirchhoff. They showed that the intensity at each point of the secondary wave is proportional to $(1 + \cos\theta)$ where θ is the angle between the wave normal and the line joining the point of the secondary wavelets and the sources S. For the point of the secondary wavelet, which is directly behind the wave, the value of $\theta = \pi$ and hence $(1 + \cos\theta)$ is zero. Thus the intensity of the secondary waves in the backward direction is zero.

4.5 INTERFERENCE OF MECHANICAL WAVES

In order to understand the phenomenon of interference of light waves clearly, let us first take an illustration of mechanical wave.

Suppose, mechanical disturbances of identical nature are being made at A and B on the surface of water in a still pond. It is shown in Fig. 4.5. Waves consisting of circular crests and troughs will move over the surface of water from A and B. Crests have been shown by circles of solid curves and trough by circles of dotted curves. From the principle of superposition, we know that the resultant displacement at a point on the surface of water is obtained by algebraically summing up the displacements produced by the two waves separately at that point. The points shown by circles in the diagram will have zero displacement. The waves are said to have interfered destructively with each other at those points. Intensity of wave there is also zero. On the other hand, the points shown by cross in the diagram will have maximum displacement because, at these points either the crest of one will combine with the crest of the other or the trough of one will combine with the trough of the other. In such a case, the amplitude of the resultant wave is twice the amplitude of either wave and the intensity is quadrupled. The waves, at those points are said to be interfere constructively with each other.

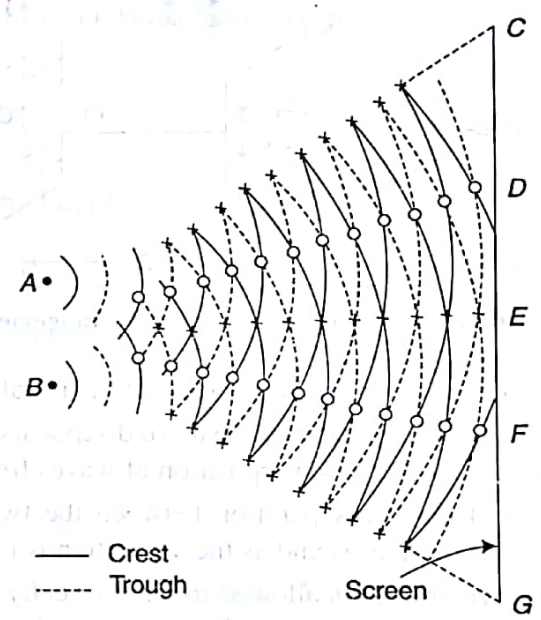


Fig. 4.5 Interference of two waves from two sources A and B.

The same thing happens in the case of light. Instead of the surface of water if we suppose that A and B are two optical sources continuously sending out identical waves in the surrounding optical medium and GEC is a screen, then we find very bright points at G, E, C, etc., and dark points at F, D, etc. (Fig. 4.5). It means that the screen will be intersected by alternate bright and dark points, known as *interference fringes*.

4.6 YOUNG'S EXPERIMENT ON INTERFERENCE OF LIGHT WAVES (DOUBLE SLIT EXPERIMENT)

The phenomenon of interference of light was first discovered by Thomas Young in 1801.

Young allowed the sun light to pass through a pin hole S Fig. (4.6) and then at some distance through two sufficiently close pin holes S_1 and S_2 in an opaque screen. Finally, the light was received on a screen on which he observed an uneven distribution of light intensity. Young found that the illumination on the screen consisted of many alternate bright and dark bands. In accordance with the modern laboratory technique,

narrow parallel slits replace pinholes and the slit S is illuminated with monochromatic light (Fig. 4.7). Light is received on a screen placed at a certain distance to the right and parallel to the plane containing the slits S_1 and S_2. According to Huygens' principle, cylindrical wavelets spread out from the slit S and as the path $SS_1 = SS_2$ the wavelets reach slits S_1 and S_2 at the same instant. Spherical waves emanating from S_1 and S_2 are coherent [see Art. 4.9 ...] and on the screen permanent interference fringes are produced. Figure 4.7 shows the section of the wave front on the plane containing S, S_1 and S_2. On the screen a number of alternate bright and dark bands of equal width, called **interference fringes**, are observed parallel to the slits. At the centre of the screen, the intensity of light is maximum.

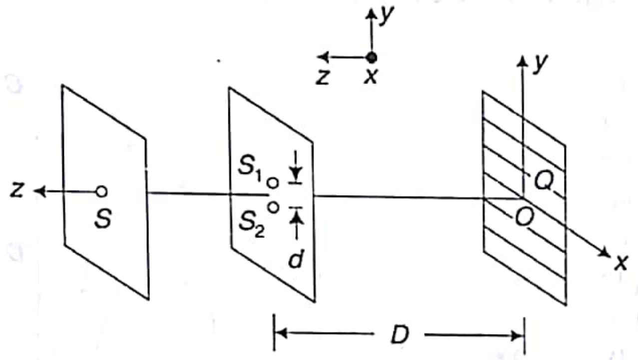


Fig. 4.6 Young's experimental arrangement.

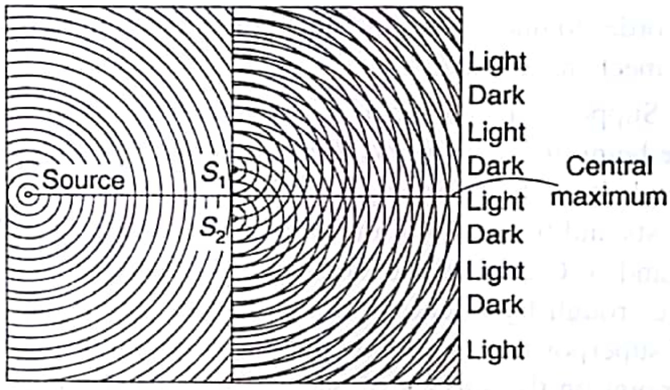


Fig. 4.7 Section of the spherical wavefront.

Young observed some characteristics of the dark and bright bands:

(i) The interference pattern disappears, if one of the slits is closed. It shows that the interference pattern is due to superposition of waves from the two slits.

(ii) As the separation between the two slits is gradually diminished the spacing between the bands increases and as the separation is increased the bands becomes narrow and ultimately disappears.

(iii) If light is allowed to enter directly through S_1 and S_2 and the slit S is removed, the bands disappear producing a general illumination on the screen.

[**Note** Young's double slit experiment is an example of interference by division of wave front. Here the wave front from S is divided at S_1 and S_2. Young explained the interference pattern by considering the principle of superposition.]

4.7 THEORY OF INTERFERENCE (ANALYTICAL TREATMENT)

Let S_1 and S_2 be two sources of monochromatic light of wavelength λ. Let two light rays from S_1 and S_2 proceed along S_1P and S_2P respectively and superpose at the point P of the medium (Fig. 4.8). From the principle of superposition, we know that the resultant displacement at P at any instant is the algebraic sum of the displacements at P due to the individual waves.

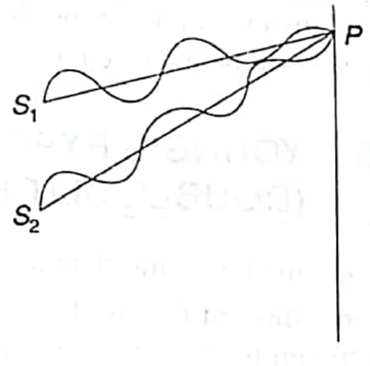


Fig. 4.8 Superposition of two monochromatic light waves

If the waves coming from sources S_1 and S_2 produce displacements y_1 and y_2 at P at an instant t, then

$$y_1 = a_1 \sin \omega t \qquad \text{...(4.1)}$$

and

$$y_2 = a_2 \sin (\omega t + \delta) \qquad \text{...(4.2)}$$

where a_1 and a_2 are the amplitudes of the waves and δ is the phase difference between the two waves. The resultant displacement at P is given by

$$y = y_1 + y_2 = a_1 \sin \omega t + a_2 \sin (\omega t + \delta)$$
$$= a_1 \sin \omega t + a_2 (\sin \omega t \cos \delta + \cos \omega t \sin \delta)$$
$$= (a_1 + a_2 \cos \delta) \sin \omega t + (a_2 \sin \delta) \cos \omega t$$

Substituting $(a_1 + a_2 \cos \delta) = A \cos \varphi$ ...(4.3)

and $a_2 \sin \delta = A \sin \varphi$ ...(4.4)

We have

$$y = A \cos \varphi \sin \omega t + A \sin \varphi \cos \omega t$$
$$= A \sin (\omega t + \varphi) \qquad ...(4.5)$$

where A and φ are new constants.

Hence, the resultant displacement at point P is a simple harmonic wave of amplitude A and phase difference φ. The amplitude A can be obtained by squaring Eqs. (4.3) and (4.4) and then adding, we get

$$(a_1 + a_2 \cos \delta)^2 + (a_2 \sin \delta)^2 = A^2 (\cos^2 \varphi + \sin^2 \varphi)$$

or,
$$A^2 = a_1^2 + a_2^2 \cos^2 \delta + 2a_1 a_2 \cos \delta + a_2^2 \sin^2 \delta$$

or,
$$A^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos \delta \qquad ...(4.6)$$

The phase difference φ can be obtained by dividing Eq. (4.4) by Eq. (4.3) and we get

$$\tan \varphi = \frac{a_2 \sin \delta}{a_1 + a_2 \cos \delta} \qquad ...(4.7)$$

The resultant intensity at P is proportional to square of the amplitude, hence we write $I \propto A^2$. Generally in arbitrary units, $I = A^2$.

So, from equation (4.6) the resultant intensity is given by

$$I = a_1^2 + a_2^2 + 2a_1 a_2 \cos \delta \qquad ...(4.8)$$

4.7.1 Condition for Constructive Interference

When $\delta = 0, 2\pi, 4\pi$, etc., $\cos \delta = 1$ which is the maximum value of the cosine function. Hence $I_{max} = a_1^2 + a_2^2 + 2a_1 a_2 = (a_1 + a_2)^2$. The two waves reinforce each other completely and the resultant is thus maximum; in general when

$$\delta = 2n\pi \qquad ...(4.9)$$

(where $n = 0, 1, 2, 3$ ...etc.) the resultant intensity becomes maximum.

In terms of optical path difference, the condition for maximum intensity is

$$S_2 P - S_1 P = 2n \times \frac{\lambda}{2} \qquad \left[\delta = \frac{2\pi}{\lambda} \times \text{path difference} \right] \qquad ...(4.10)$$

So, at all points where the path difference is an even multiple of $\frac{\lambda}{2}$, the waves will interfere constructively producing very bright points and their interference is known as **constructive interference.**

4.7.2 Condition for Destructive Interference

When $\delta = \pi, 3\pi, 5\pi$, etc., $\cos \delta = -1$ which is the least value of the cosine function. Hence $I_{min} = a_1^2 + a_2^2 - 2a_1a_2 = (a_1 - a_2)^2$. This is the minimum value of the intensity. Here the two waves interfere destructively. In general, when

$$\delta = (2n + 1)\pi \qquad ...(4.11)$$

(where $n = 0, 1, 2, 3$, ...etc.) The resultant intensity becomes minimum. In terms of the optical path difference, the condition for minimum intensity is

$$S_2P - S_1P = (2n+1)\frac{\lambda}{2} \qquad \text{...(4.12)}$$

So, at all points where the path difference is an odd multiple of $\frac{\lambda}{2}$, the waves will interfere destructively producing dark points and their interference is known as **destructive interference.**

4.8 INTENSITY DISTRIBUTION CURVE AND ENERGY CONSERVATION

A curve showing the variation of intensity with phase is called the intensity distribution curve. The same is shown in Fig. (4.9).

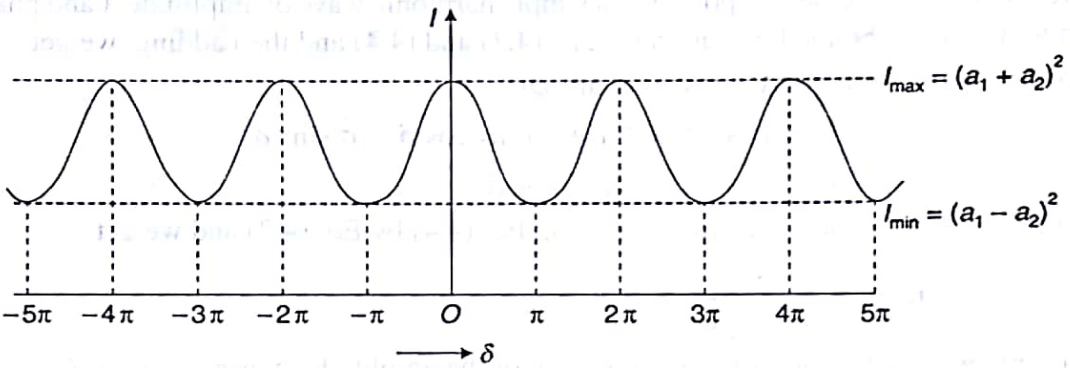


Fig. 4.9 Intensity distribution curve of interference.

When two light waves interfere, darkness is produced at some points. It appears that energy is destroyed at the points of destructive interference and created at the points of constructive interference. So, the conservation principle of energy appears to be violated. But this not correct. The energy is merely redistributed between maxima and minima. The energy is only transferred from the region of destructive interference to the region of constructive interference. The average value of the energy over any number of fringes is the same as it would be if the interference effects were absent. For example, the average value of the intensity on the screen over the range, $\delta = 0$ to $\delta = 2\pi$ is given by

$$I_{average} = \frac{\int_0^{2\pi} I\, d\delta}{\int_0^{2\pi} d\delta} = \frac{\int_0^{2\pi} (a_1^2 + a_2^2 + 2a_1a_2\cos\delta)\, d\delta}{\int_0^{2\pi} d\delta}$$

$$= \frac{\left[a_1^2\,\delta + a_2^2\,\delta + 2a_1\,a_2\sin\delta\right]_0^{2\pi}}{[\delta]_0^{2\pi}}$$

$$= \frac{(a_1^2 + a_2^2)\,2\pi}{2\pi} = a_1^2 + a_2^2 = I_1 + I_2$$

This means that the energy of the maxima is increased at the expense of the energy of the minima. And the averaged value of the intensity equals the sum of saperate intensities. Thus, in the phenomena of interference the energy is conserved.

4.9 CONDITIONS FOR PERMANENT INTERFERENCE: COHERENT SOURCES

The interference pattern in which the positions of maximum and minimum intensity of light remain fixed along the screen is called **sustained or permanent interference pattern.**

For producing sustained interference, following conditions must be satisfied:

(i) The two sources must be coherent. It means that the two sources should emit waves in the same phase or should have a constant phase difference between them. Two completely independent sources will not form coherent sources because their individual phases change in a completely arbitrary way.

(ii) The sources should emit light of same wavelength and same or nearly same amplitude.

(iii) The sources should be narrow and close to each other to observe distinct fringes.

(iv) At maxima, the path difference between the waves should be even multiple of $\frac{\lambda}{2}$ and at minima odd multiple of $\frac{\lambda}{2}$.

(v) The two light waves must be in the same state of polarisation.

(vi) The two waves must propagate along the same direction to get coincidence.

4.10 CLASSIFICATION OF INTERFERENCE PHENOMENA

There are two practical methods of obtaining two coherent sources from a single source.

A. *Division of Wavefront* In this method we require a point source or a line source, from which light waves spread out as spherical or cylindrical waves. Any two points on such a spherical or cylindrical wave front are taken as the interfering sources, which is evidently coherent. Fresnel's bi-prism, Lloyd's mirror, Billet's split produce two coherent sources by the above method.

B. *Division of Amplitude* In this method we require an extended source. Devices which divide the amplitude of the incoming wave of light into two or more parts by partial reflection and refraction and thereby give rise to two or more beams which are later made to reunite to produce the interference effects, come under this class. Some of the examples of this class are Michelson's interferometer, Newton's ring experimental set up, Fabry–Perot interferometer, etc.

4.11 CALCULATION OF FRINGE WIDTH

Suppose A and B are two monochromatic coherent sources of light originating from S sending out waves of same amplitude and same wavelength keeping same phase relationship between them (Fig. 4.10). Let $AB = 2d$ and O is the midpoint of AB. The distance of the screen from O is $OC = D$. Now C is equidistant from A and B. So the waves from A and B will meet in same phase at C. So the light intensity will be maximum at C. **It is called central maximum point.**

Now, let us take another point P on the screen at a distance x from C. Join AP and BP. Since the paths are unequal, the phase of the waves arriving at P from A and B will be different. So, whether the intensity at P will be maximum or minimum that depends on the path difference $(BP - AP)$.

Now, $$BP^2 = D^2 + (x+d)^2 \quad ...(4.13)$$

and $$AP^2 = D^2 + (x-d)^2 \quad ...(4.14)$$

$\therefore$ $$BP^2 - AP^2 = (x+d)^2 - (x-d)^2 = 4xd$$

or, $$(BP + AP)(BP - AP) = 4x \cdot d$$

$$BP - AP = \frac{4xd}{BP + AP} = \frac{4xd}{2D} = \frac{2xd}{D}$$

[The point P lies very close to C and D is very large compared to $2d$. So $BP \simeq AP \simeq D$]

The path difference of the waves arriving at P is given by

$$BP - AP = \frac{2xd}{D} \quad ...(4.15)$$

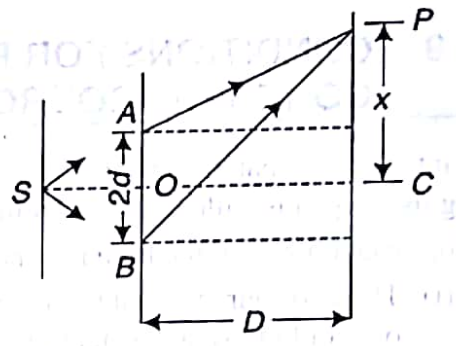


Fig. 4.10 Diagram of double slit experiment

(a) *Bright fringes* Now, if P is the position of nth bright fringe then path difference is $\frac{2x_n d}{D}$ where x_n is the distance of the nth bright fringe from central maximum.

Now for bright fringe, the path difference should be equal to even integral multiple of $\frac{\lambda}{2}$. So

$$\frac{2x_n d}{D} = 2n\frac{\lambda}{2}$$

or, $$x_n = \frac{Dn\lambda}{2d} \quad ...(4.16)$$

Now for $(n + 1)$th bright fringe, from Eq. (4.16), we can express the distance of that point from central maximum as

$$x_{n+1} = \frac{D(n+1)\lambda}{2d} \quad ...(4.17)$$

Therefore, the width of the fringe in case of bright bands, i.e., the distance between two consecutive bright bands is

$$\beta = \frac{D(n+1)\lambda}{2d} - \frac{Dn\lambda}{2d} = \frac{D\lambda}{2d} \quad ...(4.18)$$

(b) *Dark fringes* If P is the position of nth dark fringe, then path difference should be equal to odd multiple of $\lambda/2$. So

$$\frac{2x_n d}{D} = (2n+1)\frac{\lambda}{2}$$

or, $$x_n = \frac{(2n+1)D\lambda}{4d} \quad ...(4.19)$$

Now for $(n + 1)$th dark fringe, from Eqs. (4.19), the distance of that point from central maximum is

$$x_{n+1} = \frac{D[2(n+1)+1]\lambda}{4d} \quad ...(4.20)$$

Therefore, the fringe width in case of dark bands i.e. the distance between two consecutive dark is

$$\beta' = \frac{D[2(n+1)+1]\lambda}{4d} - \frac{D\lambda}{4d}(2n+1) = \frac{D\lambda}{2d} \quad ...(4.21)$$

So from Eqs. (4.18) and (4.21) we see that the distance between two consecutive bright or dark bands is independent of the order of the fringe (i.e., n). Hence all bright bands and all dark bands have equal width. The fringe width (β), varies directly with D the slit screen separation, inversely with the separation of slits $2d$ and directly with the wave length λ of light employed.

4.12 SHAPE OF INTERFERENCE FRINGES

In the preceding article, we have seen $BP^2 = D^2 + (x + d)^2$ and $AP^2 = D^2 + (x - d)^2$.

So the path difference $\Delta = BP - AP = [D^2 + (x + d)^2]^{1/2} - [D^2 + (x - d)^2]^{1/2}$

or, $\Delta + [D^2 + (x - d)^2]^{1/2} = [D^2 + (x + d)^2]^{1/2}$

On squaring both sides,

$$\Delta^2 + 2\Delta\,[D^2 + (x - d)^2]^{1/2} + D^2 + (x - d)^2 = D^2 + (x + d)^2$$

or, $2\Delta\,[D^2 + (x - d)^2]^{1/2} = 4xd - \Delta^2$

On squaring both sides, this equation easily reduces to,

$$4(4d^2 - \Delta^2)\,x^2 - 4\Delta^2 D^2 = \Delta^2\,(4d^2 - \Delta^2)$$

or,
$$\frac{x^2}{\Delta^2/4} - \frac{D^2}{(4d^2 - \Delta^2)/4} = 1 \qquad \text{...(4.22)}$$

This is the equation of hyperbola in a standard form. The eccentricity (e) equal to

$$e = \left[\frac{\Delta^2}{4} + \frac{4d^2 - \Delta^2}{4}\right]^{1/2} \Big/ \left(\frac{\Delta}{2}\right) \simeq \frac{2d}{\Delta} \qquad \text{...(4.23)}$$

Since Δ is very small, the eccentricity is very large. Hence the fringes appear as straight lines in the region of observation.

4.13 COHERENCE

The coherence of a wave describes the accuracy with which it can be represented by a pure sine wave. **If a wave appears to be pure sine wave for an infinitely extended time or an infinitely extended space, it is called a perfectly coherent wave.** The characteristic of such a wave is that a definite relationship between the phases of the wave at a certain time and at a given time later, or at a certain space point and at another given space-point exists. A perfect coherence is however, an idealization. A real light source always emits light in short pulses. Light waves that are pure sine waves only in a limited space or for a limited time are only partially coherent waves. There are two different criteria of coherence. The first criterion is related to the concept of temporal coherence which expresses the correlation to be expected between a wave at a given time and a certain time later. The second criterion is related to the concept of spatial coherence which expresses the correlation to be expected between a wave at a given point in space and a point at a certain distance away. Laser light is characterized by a high degree of ordering of the light field than the other sources. Due to its coherence only, it is possible to create high power (10^{13} W) in space with laser beam only 1 μm in diameter.

4.13.1 Temporal Coherence

Temporal coherence refers to the correlation between the light field at a space point at a time and that at the same space point at a later time. The oscillating electric field E of a perfectly coherent light wave has a constant amplitude at any point in space, but its phase will vary with time linearly. As a function of t, the field

E will appear as illustrated in Fig. (4.11). The displacement of the ideal electric field E remains sinusoidal for $-\infty < t < +\infty$ and does not agree with the real cases.

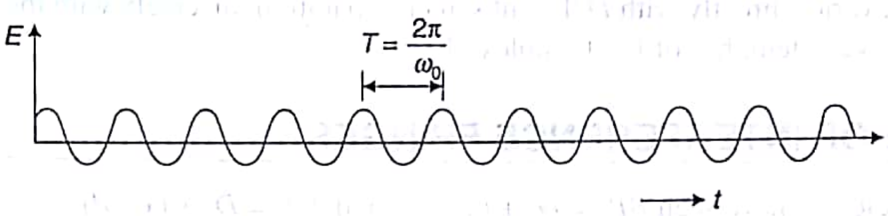


Fig. 4.11 A wave train of infinite length. Displacement remains sinusoidal for $-\infty < t < +\infty$.

An actual source (real source) can never produce an ideal sinusoidal field for all time. When an excited atom makes a transition back to its initial state, it emits a light pulse of extremely short duration ~10^{-10} s. The field therefore is sinusoidal for an interval of ~10^{-10} s. The light pulse due to this transition is not a continuous wave of infinite extent but a wave of finite extent. It is known as wave packet. The light from a source is a bundle of such wave packet originating from different atoms. Different atoms of the source emit radiation in a random fashion. Every wave packet propagates with a sustainable phase for time duration of 10^{-10} s after which it has a random phase. Figure 4.12 illustrates the field due to a real source of light. In Fig. (4.12), τ_c represents the average duration of the wave trains i.e., the electric field remains sinusoidal for times of the order of τ_c. *The average time-interval for which the 'field' of light vector remains sinusoidal, and therefore a definite phase relationship exists, is termed as coherence time,* τ_c (or Δt).

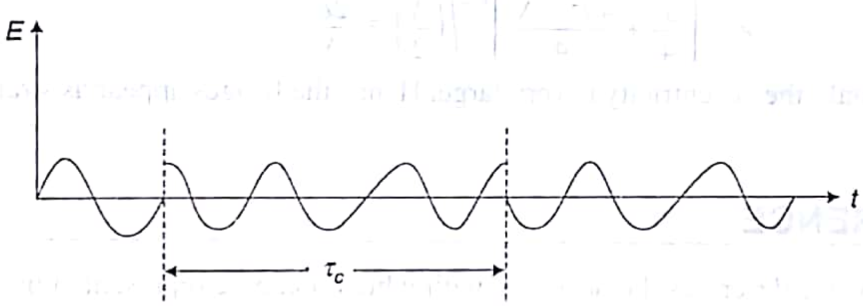


Fig. 4.12 Coherent time for an actual source.

The distance over which the field is sinusoidal is termed the coherent length l_c of the light beam and is *given by*

$$l_c = \tau_c \times c \qquad \text{...(4.24)}$$

where c is the speed of light in free space. For example, for the Ne line ($\lambda = 6328$ Å), $\tau_c \sim 10^{-10}$ s and for the red Cd line ($\lambda = 6438$ Å) $\tau_c \sim 10^{-9}$ s; the corresponding coherence lengths are 3 cm and 30 cm respectively.

The temporal coherence τ_c of the light beam is related to the spectral width $\Delta\lambda$ given by

$$\tau_c = \frac{\lambda^2}{\Delta\lambda} \qquad \text{...(4.25)}$$

and coherence length

$$l_c = \frac{\lambda^2}{\Delta\lambda} c \qquad \text{...(4.26)}$$

In terms of frequency, coherent length l_c can be expressed as

$$l_c = \frac{c}{\Delta\nu} \qquad \text{...(4.27)}$$

where $\Delta\nu$ is the line width.

The light emitted by a laser is more monochromatic (ideal harmonic wave of infinite extent because $\Delta\nu$ is equal to zero) than that of any conventional monochromatic source. The degree of non-monochromaticity (ξ) of a wave may be defined as its relative band width and is given by $\xi = \frac{\Delta\nu}{\nu}$

For a highly stable gas laser $\Delta\nu = 500$ Hz

and $\nu = 5 \times 10^{14}$ Hz

$\therefore$ the degree of non-monochromacity $= \dfrac{500}{5 \times 10^{14}} = 10^{-12}$

But for conventional monochromatic source, the degree of non-monochromacity $= \dfrac{10^9}{10^{14}} = 10^{-5}$

Therefore laser source is more monochromatic than conventional source.

4.13.2 Spatial Coherence

The spatial coherence refers to the correlation between two light fields at two different space points at an instant of time on a wave front of a given light wave. If two light fields at two different points in space preserve a constant phase difference over any time t then they are said to have spatial coherence.

Let the light waves be emitted from the source S (Fig. 4.13) and A and B be two space points on a line joining them with S. The phase relationship between the points A and B depends on (i) the distance AB and (ii) the temporal coherence of the beam.

If $AB << l_c$, the coherent length, then there will be a definite phase relationship between A and B. So, there will be a high degree of coherence between the points A and B. Conversely, if the distance $AB >> l_c$ the coherent length, then there will be no coherence between the points A and B.

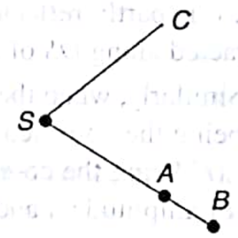


Fig. 4.13 Spatial coherence

We now consider the points A and C which are equidistant from the source S, but unlike the previous case, not on the same line joining with S. If S is true point source, light waves will reach A and C exactly in the same phase. In other words, the two space points will be in perfect spatial coherence. In case the source is an extended one, the two points A and C will no longer be in coherence. For extended source, it is not necessary that phase at two points at equal optical path distance would be equal because optical path for different waves emitted from different parts of the source would be different. Therefore, **spatial coherence depends also on size of the source.** The lateral coherence width (l_w) of an extended incoherent source represents the distance over which the beam may be assumed to be spatially coherent; it is given by

$$l_w = \frac{\lambda}{\theta} \qquad \text{...(4.28)}$$

where θ is the angle substanded by the source at the point of observation.

In the case of interference of light, the degree of contrast of fringes depends on the degree of spatial coherence of interfering waves. The contrast of interference fringes varies as the optical path difference (Δ) is varied, beginning from an extremely good contrast for $\Delta << l_c$ to a very poor contrast for $\Delta >> l_c$.

The spatial coherence is directly related to the degree of directionality of light waves. For higher degree of spatial coherence, higher would be the directionality. Since laser beam is highly directional one, so it is more spatially coherent than ordinary light.

So we may conclude that degree of contrast of fringe pattern is a measure of degree of spatial coherence and monochromaticity is a measure of temporal coherence.

4.14 CHANGE OF PHASE DUE TO REFLECTION: STOKES' LAW

Stokes' law states that if waves are reflected at a rarer to denser medium interface (for example air-glass interface), the reflected waves suffer a phase change of π $\left(\text{or path difference } \frac{\lambda}{2}\right)$ compared to incident waves.

But if reflection takes place at a denser to rarer medium interface, then no change of phase or path difference takes place. In Fig. (4.14) AB is the surface separating the denser medium below it from a rarer medium above it. A light wave of amplitude a is incident at O in the direction PO on the surface of separation. A part of the light is reflected along OR and the remaining part is refracted into the denser medium along OT. If r and t be the reflection and transmission co-efficients respectively, then the amplitude of reflected wave OR is ar and the amplitude of the refracted wave along OT is at. If there be no loss of energy, then $r + t = 1$. Now if the reflected and refracted rays OR and OT be made to retrace their path, then they should combine along OP to yield the original amplitude a.

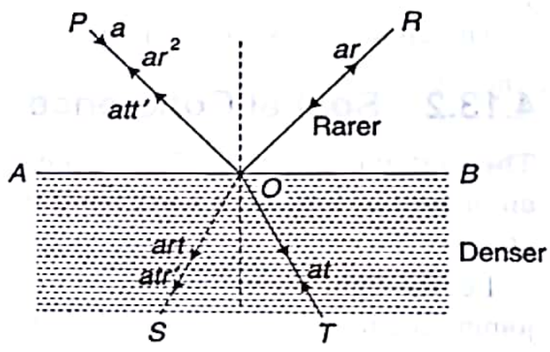


Fig. 4.14 Stoke 's' treatment.

If the ray OR of amplitude ar is made to retrace its path along RO, it is partly reflected along OP with amplitude ar^2 and partly refracted along OS of amplitude art.

Similarly, when the ray TO of amplitude at is reversed, it is partly refracted along OP with amplitude att' (t' being the co-efficient of transmission in the rarer medium) and partly reflected along OS with amplitude atr' (r' being the co-efficent of reflection in the denser medium). Since originally we had an incident wave PO of amplitude a and no wave along OS, we must have

$$ar^2 + att' = a \qquad \ldots(4.29)$$

$$art + atr' = 0 \qquad \ldots(4.30)$$

$\therefore$ from Eq. (4.29), we get $\quad tt' = 1 - r^2$

From Eq. (4.30), we get $\quad r' = -r$

The negative sign implies a displacement in opposite direction that is equivalent to a phase change of π or a path difference of $\frac{\lambda}{2}$.

Hence, the two rays one reflected from a denser medium and the other reflected from a rare medium differ in phase by π from each other.

4.15 INTERFERENCE IN A THIN FILM (REFLECTED LIGHT)

When white light is reflected by thin films like soap bubbles, oil layers on water and oxide layers on metal surfaces a variety of colours can be seen. This is due to interference between light reflected by the front and back surfaces of these films.

Here we, consider a thin film of thickness t and refractive index μ bounded by two plane surfaces AB and $A'B'$ (Fig. 4.15]. Let, the light wave from point source P be incident at O at an angle i. A part of the light is reflected along OT and the other part OQ is refracted into the film towards Q making angle of refraction. At Q, it is again partly reflected along QR inside the medium and partly refracted out of the medium along QC. The ray QR finally emerges out along RS.

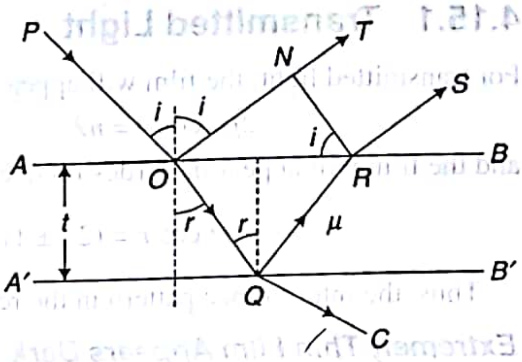


Fig. 4.15 Waves reflected and transmitted by a thin film.

The rays OT and RS are originated from the same source. So, if they meet at a point will produce interference. But there is a path difference between the two rays.

To calculate the additional path travelled by RS inside the film, drop a perpendicular RN on OT. Hence, the path difference between the rays RS and NT is

$$\delta' = \text{optical path } OQR \text{ inside the film} - ON$$

$$= \mu\,(OQ + QR) - ON \quad \text{...(4.31)}$$

$$= 2\mu\, OQ - ON \quad [\because OQ = QR]$$

$$= 2\mu\, OQ - OR \sin i$$

Now $\quad t = OQ \cos r \quad$ or, $\quad OQ = \dfrac{t}{\cos r} \quad$ and $\quad OR = 2t \tan r$

$$\therefore \quad \delta' = 2\mu \frac{t}{\cos r} - 2t \tan r \sin i$$

$$= \frac{2\mu t}{\cos r}(1 - \sin^2 r) = 2\mu t \cos r \quad \text{...(4.32)}$$

Now, an additional path difference of $\dfrac{\lambda}{2}$ is introduced in the path of the ray OT as it is reflected from the surface of an optically denser medium.

Hence, the effective path difference between the two reflected rays

$$\delta = 2\mu\, t \cos r \pm \frac{\lambda}{2} \quad \text{...(4.33)}$$

If the path difference be $\delta = n\lambda$ where $n = 0, 1, 2, 3 \ldots$

Then *constructive interference* takes place and the film appears bright.

$$\therefore \quad 2\mu t \cos r \pm \frac{\lambda}{2} = n\lambda$$

$$\text{or,} \quad 2\mu t \cos r = (2n \mp 1)\frac{\lambda}{2} \quad \text{...(4.34)}$$

If the path difference be $\delta = (2n \pm 1)\dfrac{\lambda}{2}$ where $n = 0, 1, 2, \ldots$ *destructive interference* takes place and the film appears *dark*.

$$\therefore \quad 2\mu t \cos r \pm \frac{\lambda}{2} = (2n \pm 1)\frac{\lambda}{2}$$

$$\text{or,} \quad 2\mu t \cos r = n\lambda \quad \text{...(4.35)}$$

4.15.1 Transmitted Light

For transmitted light, the film will appear bright (constructive interference) when

$$2\mu t \cos r = n\lambda \qquad \text{...(4.36)}$$

and the film will appear dark (destructive interference) when

$$2\mu t \cos r = (2n \pm 1)\frac{\lambda}{2} \qquad \text{...(4.37)}$$

Thus, the interference pattern in the reflected and transmitted system are complementary to each other.

Extremely Thin Film Appears Dark When the thickness of the film is very small compared to the wave length ($t << \lambda$), so that $2\mu t \cos r$ is negligible, the two rays will be out of phase $\left(\text{path difference} = \frac{\lambda}{2}\right)$ and therefore the film will appear dark.

4.15.2 Colouring Effect in Thin Film

If white light (e.g., sun light) is incident on the film (oil film floating on water) instead of monochromatic light, a beautiful colouring effect is observed on the film. White light contains seven visible colours of different wavelengths. As a result, if destructive interference occurs for one colour, it will not be true for all colours. Hence excepting one colour, all other colours will be seen by the eyes. Consequently, the oil film floating on water will show a variety of colours.

4.15.3 No Colouring Effect in Thick Film

If a film has large thickness, then at a given point, the condition of brightness will be satisfied for large number of wave lengths. Hence no particular colour will finally appear at the given point. Thus, a thick film exhibits no colour effect.

4.15.4 Necessity of Broad Source of Light in Thin Film

To produce coherent sources by the method of division of wave front (Fresnel bipsipm, Lloyd's mirror) the source of light is taken in the form of narrow slit. But to produce coherent sources by the method of division of amplitude, we require extended source.

When a thin film is illuminated with light from a point source and it is observed with a lens of small apperture such as the eye, the ray that can enter the eye are confined to a small range of directions. But in case of broad source the rays incident at different angles on the film are accommodated by the eye and the field of view is large. Due to this reason to observe interference phenomena in thin film, a broad source is required.

4.16 INTERFERENCE IN WEDGE SHAPED FILM

A wedge-shaped air film is enclosed by two plane surfaces inclined at an angle α. If the film is illuminated by a monochromatic light PO, then the reflected ray OT from the front surface and the other emergent ray ST reflected from the back surface BC of the film will interfere (Fig. 4.16). There two rays are originated from the same source P and hence they will produce interference pattern. Let μ be the refrective index of the material of the film. The path difference between the two rays OT and ST is given by,

$$\delta' = \text{optical path } ODS \text{ inside the film} - ON$$
$$= \mu(OD + DS) - ON$$
$$= \mu(OD' + D'D + DS) - ON$$
$$= \mu(D'D + DS) \quad [\because ON = \mu OD'] \quad \ldots(4.38)$$

Draw SME $\perp$ BC. Produce OD and SM to meet at E. From the geometry of Fig. (4.16) $DE = DS$ and $SM = ME = t$ = the thickness of the film at S.

$$\therefore \quad \delta' = \mu(D'D + DE) = \mu D'E = 2\mu t \cos(r - \alpha) \quad \ldots(4.39)$$

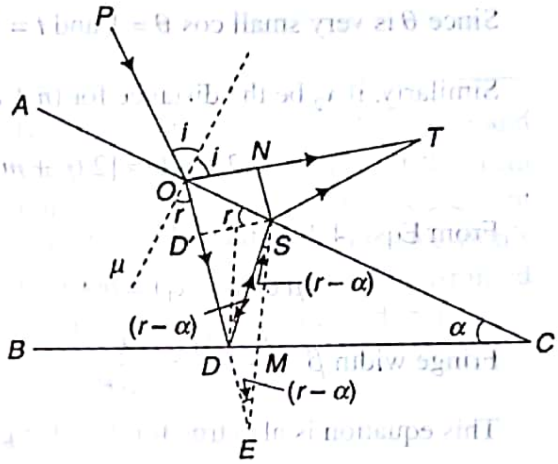


Fig. 4.16 Interference in wedge-shaped film.

In addition to this path difference, there is an extra phase difference of π, or path difference $\pm\frac{\lambda}{2}$ caused by reflection at O, from the surface backed by denser medium. The reflection at D, from the surface backed by rarer medium, will not cause any change of phase.

Hence the total path difference between the two reflected rays is,

$$\delta = \delta' \pm \frac{\lambda}{2} = 2\mu t \cos(r - \alpha) \pm \frac{\lambda}{2} \quad \ldots(4.40)$$

For constructive interference

$$2\mu t \cos(r - \alpha) \pm \frac{\lambda}{2} = \text{even multiple of } \frac{\lambda}{2}$$

or, $$2\mu t \cos(r - \alpha) = \text{odd multiple of } \frac{\lambda}{2}$$

$$= (2n \mp 1)\frac{\lambda}{2} \quad \text{where } n = 0, 1, 2, 3, \quad \ldots(4.41)$$

For destructive interference

$$2\mu t \cos(r - \alpha) \pm \frac{\lambda}{2} = \text{odd multiple of } \frac{\lambda}{2}$$

or, $$2\mu t \cos(r - \alpha) = \text{even multiple of } \frac{\lambda}{2}$$

or, $$2\mu t \cos(r - \alpha) = n\lambda \quad \text{where } n = 0, 1, 2, 3 \quad \ldots(4.42)$$

4.16.1 Fringe Width

We take a point at a distance x_1 from O, where thickness of the film t (Fig. 4.17). For normal incidence (i.e., $r = 0$) we have

$$2\mu t \cos\theta = (2n + 1)\frac{\lambda}{2}$$

for constructive interference ...(4.43)

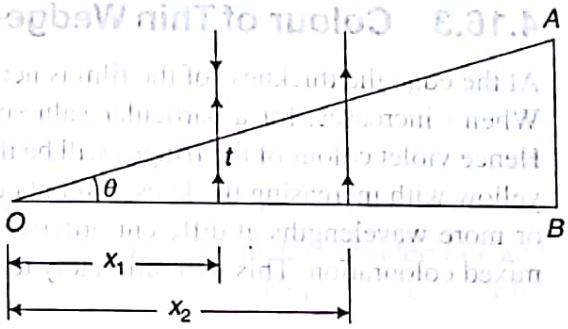


Fig. 4.17

Since θ is very small $\cos\theta = 1$ and $t = x_1\,\theta$. Equation (4.43) reduced to $2\mu\, x_1\,\theta = (2n+1)\dfrac{\lambda}{2}$. ...(4.44)

Similarly, if x_2 be the distance for $(n + m)$ bright band, then

$$2\mu\, x_2\,\theta = [2\,(n+m)+1]\frac{\lambda}{2} \qquad \text{...(4.45)}$$

From Eqs. (4.44) and (4.45)

$$2\mu\,\theta\,(x_2 - x_1) = m\lambda \qquad \text{...(4.46)}$$

Fringe width $\beta = \dfrac{x_2 - x_1}{m} = \dfrac{\lambda}{2\mu\theta}$ ...(4.47)

This equation is also true for dark fringe

Again $\qquad (x_2 - x_1)\,\theta = \dfrac{m\lambda}{2\mu}$

or, $\qquad t_2 - t_1 = \dfrac{m\lambda}{2\mu}$ ...(4.48)

It gives the difference of film thickness between two points.

4.16.2 Discussions

(i) When the thickness of the film is very small compared to the wavelength, i.e., $t << \lambda$ then $2\mu t\cos r$ is negligible in comparison to λ and we have $\delta \approx 0$. This condition corresponds to zero intensity in the reflected part but maximum intensity for transmitted part

(ii) Classification of fringes: We have $\delta = 2\mu t\cos r$, from this we see that δ depends upon (i) λ, (ii) μt and (iii) r.

Therefore, for variation of each of them there are different types of fringes.

(a) When μt and r are kept constant the fringe system depends upon λ, i.e., on the colour of the light. These are known as the fringes of equal chromatic order (FECO).

(b) When λ and r are kept constant, we have fringe system dependent upon μt. These are called fringes of equal thickness or iso-pachic fringes. These are localized fringes.

(c) When λ and μt are kept constant, the fringe system depends upon r. These are known as fringes of equal inclination or iso-clinic fringes. These fringes are localised at infinity. The fringes of equal inclination is sometimes also called Haidinger's fringes.

4.16.3 Colour of Thin Wedge-Shaped Film

At the edge the thickness of the film is negligible. Therefore, the edge will appear dark under reflected light. When t increases, for a particular value of t brightness condition will be satisfied first by the violet light. Hence violet colour of the fringes will be the first one. Then the fringe of other colour follow, i.e., blue, green, yellow with increasing thickness. But at certain thickness the condition for constructive interference for two or more wavelengths at different orders may be satisfied. Hence the regions of greater thickness will have mixed colouration. This will ultimately terminate into uniform colouration.

4.17 NEWTON'S RING

An air film of gradually increasing thickness outward is formed between an optically plane glass plate and curve surface of a plano-convex lens. The thickness of the film at the point of contact is zero. If this film is illuminated by a broad-source of monochromatic light a ring system alternately bright and bark are seen around the point of contact. These rings were first discovered by Newton, that is why they are called Newton's rings. In general, in the reflected light the ring system starts with dark ring at the centre and for transmitted light it starts with bright ring at the centre. Typical Newton's rings are shown in Fig. [4.18(a) and (b)]

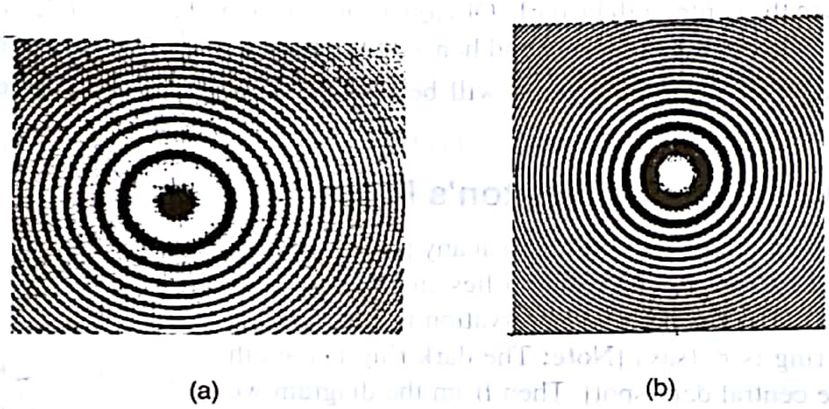


Fig. 4.18 (a) Newton's rings in reflected light (b) Newton's rings in transmitted light.

The adjoining diagram (Fig 4.19) shows a sectional view of the Newton's ring apparatus. A light ray AB falls on the film surface and a portion gets reflected along BD.

Another portion BC is transmitted through the film and partially reflected from the glass plate along CB_1 and ultimately follows the path $B_1 D_1$. There is transmitted ray CT. It is obvious that there is reflection also at B_1 which produces the next transmitted ray $C_1 T_1$. Thus, BD and B_1D_1 are two coherent beams which may interfere. Similarly, transmitted rays CT and $C_1 T_1$ may also interfere. This is because all the rays are obtained from the same incident beam by the division of amplitude. Therefore they are coherent beams.

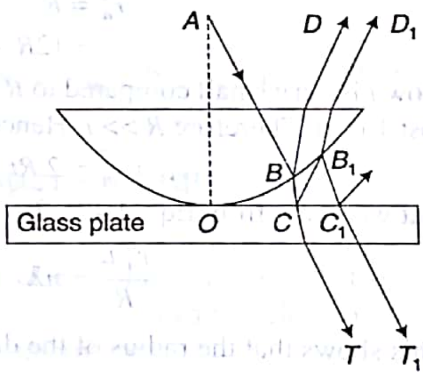


Fig. 4.19 A sectional view of the Newtons rings apparatus.

For reflected light the condition of interference are,

$$2\mu t \cos (r - \alpha) = 2n \frac{\lambda}{2} \qquad \text{for darkness.}$$

$$2\mu t \cos (r - \alpha) = (2n + 1) \frac{\lambda}{2} \qquad \text{for brightness}$$

Here μ is refractive index of the thin film, t is the thickness of the film at the point under consideration and α is the angle between the surfaces at the same point. Here we have also considered the phase change π for the ray $B_1 D_1$ due to reflection from the denser medium, i.e., the glass plate.

Generally light is made to fall normally and the radius of curvature of the lens is very large and hence we can rewrite the above conditions as

$$2\mu t = n\lambda \quad \text{for minima} \qquad \text{...(4.49)}$$

$$2\mu t = (2n+1)\frac{\lambda}{2} \quad \text{for maxima} \qquad \text{...(4.50)}$$

For transmitted light as there is no extra phase change π, the conditions for maxima and minima are just the reverse.

As the shape of the film is symmetrical with respect to the touching point it is obvious that a constant thickness (t) will be on the circumference of a particular circle with centre at the touching point. Therefore, the ring system will be concentric circles. When $t = 0$, i.e., at the centre, the reflected interfering beams will be out of phase and hence the centre will be dark. Obviously, next ring will be bright due to the relation for the condition of brightness. This will be repeated and hence the ring system will be alternately bright and dark.

The ring system with the transmitted beam will be complementary to that of reflected system due to absence of extra phase difference π.

4.17.1 Working Formula for Newton's Rings

Figure 4.20 shows that the thickness t of the film at any point is related to the radius of the circular ring BD on which it lies and the radius of the curvature of the lens R. If the point of observation is dark and the radius of the nth dark ring is r_n (say) [**Note:** The dark ring is the nth dark ring excluding the central dark spot]. Then from the diagram we have,

$$r_n^2 = R^2 - (R-t)^2$$

$$= (2R - t)t \qquad \text{...(4.51)}$$

Now t is very small compared to R (R is the order of 100 cm and t is almost 1 cm). Therefore $R >> t$. Hence

$$r_n^2 = 2\,Rt \qquad \text{...(4.52)}$$

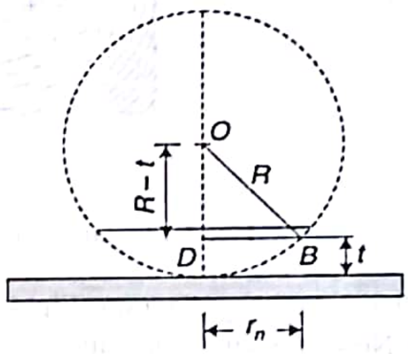


Fig. 4.20 Determination of radii of Newtons rings

But we know from Eq. (4.49), $2\mu t = n\lambda$ for dark fringe

$$\therefore \quad \frac{r_n^2 \mu}{R} = n\lambda \quad \text{or,} \quad r_n^2 = \frac{Rn\lambda}{\mu} \qquad \text{...(4.53)}$$

This shows that the radius of the dark ring is proportional to the square root of natural number.

The corresponding diameter is obviously given by,

$$d_n^2 = (2r_n)^2 = \frac{4Rn\lambda}{\mu} \qquad \text{...(4.54)}$$

This is the expression for the diameter of any nth dark ring. Similarly, we can show the diameter of any nth bright ring is given by

$$d_n^2 = \frac{2(2n+1)\,R\lambda}{\mu} \qquad \text{...(4.55)}$$

Obviously, the expression for the diameter of the $(n+p)$th dark ring is given by,

$$d_{n+p}^2 = \frac{4(n+p)\,R\lambda}{\mu} \qquad \text{...(4.56)}$$

So, from Eqs. (4.54) and (4.56) we have

$$\lambda = \frac{(d_{n+p}^2 - d_n^2)\,\mu}{4pR} \qquad \text{...(4.57)}$$

Thus we can find out λ, i.e., the wave length of a monochromatic light if we can measure the diameters of different rings. Same result can be obtained for bright rings by considering d_{n+p} and d_n the diameters of $(n + p)$th and nth rings.

4.17.2 Newton's Rings in Transmitted Light

For transmitted light the central spot is bright and for nth bright rings

$$2\mu t = n\lambda$$

Again, $t = \frac{r_n^2}{2R}$ where r_n is the radius of the n^{th} bright ring.

Hence, for nth bright ring

$$2\mu\left(\frac{r_n^2}{2R}\right) = n\lambda$$

or,

$$r_n^2 = \frac{n\lambda R}{\mu}$$

For nth dark ring

$$r_n^2 = (2n+1)\frac{\lambda R}{2\mu} \qquad \text{...(4.58)}$$

When $n = 0$, for bright ring $r_n = 0$

Therefore, in the case of Newton's rings for transmitted light, the central ring is bright (Fig. 4.18b) which is just opposite to the rings pattern produced by reflected light.

4.17.3 Refractive Index (R.I.) Calculation of Transparent Liquid by Newton's Rings Method

A liquid film can be produced instead of air film by putting few drops of experimental liquid between the plano-convex lens and the glass plate. Then by using a monochromatic light of known wavelength, the diameters of nth and $(n + p)$th dark rings can be measured by the travelling microscope.

If d'_n and d'_{n+p} be the diameters of nth and $(n + p)$th dark rings respectively then from Eq. (4.57)

$$d'^2_{n+p} - d'^2_n = \frac{4pR\lambda}{\mu} \qquad \text{...(4.59)}$$

and for air film ($\mu = 1$)

$$d_{n+p}^2 - d_n^2 = 4pR\lambda \qquad \text{...(4.60)}$$

So,

$$\mu = \frac{d_{n+p}^2 - d_n^2}{d'^2_{n+p} - d'^2_n} \qquad \text{...(4.61)}$$

4.17.4 Experimental Arrangement

To observe the Newton's ring, the experimental arrangement is made as shown in Fig. (4.21).

An air film is illuminated by sending light of single wavelength from a broad source. The light is made parallel with the help of a lens and then directed normal to the surface of the Newton's ring apparatus by the help of a plane glass plate held at 45° to the rays from the lens. The reflected light is seen by a travelling microscope held over the glass plate. The diameters of the ring system can be measured with this travelling microscope. Using Eq. (4.57) [$\mu = 1$ for air] one can easily determine the value of R for a light of known wavelength, say sodium light. Then with this determined value of R, the wavelength (λ) of any unknown light can be determined by the relation (4.57)

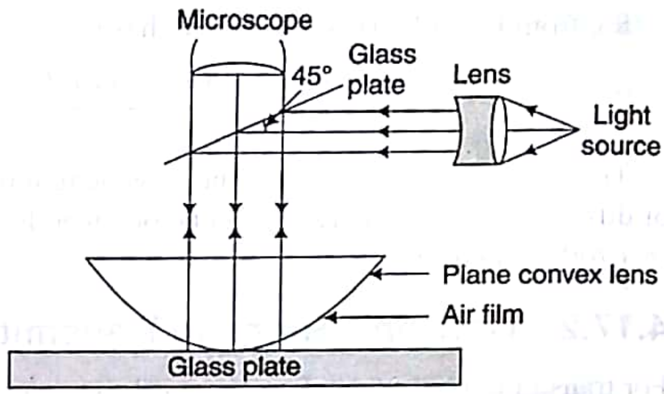


Fig. 4.21 Experimental set-up to observe Newton's rings.

4.17.5 A Few Points in Connection with Newton's Rings

(i) *Newton's rings with bright centre due to reflected light* If we place a liquid of refractive index μ between the plano-convex lens and glass plate such the $\mu_1 > \mu > \mu_2$ where μ_1 and μ_2 are the refractive indices of lens and glass plate respectively. Then the glass plate is no longer a denser medium with respect to the transparent liquid. Hence, there is no extra phase change π between the reflected interfering rays. Therefore the centre of the ring system will be bright.

The central spot will also be bright under reflected ray if $\mu_1 < \mu < \mu_2$. In this case path difference $\frac{\lambda}{2}$ take place at both surfaces.

(ii) *Disappearance of Newton's ring* Newton's rings will disappear if the glass plate is replaced by a plane mirror because the intensity of light reflected by the plane mirror will be much more compared to the intensity of light reflected by the curved surface of the lens. Under this condition, the two reflected beams will produce no interference fringe.

(iii) *White light effect* For monochromatic light, Newton's rings are alternately dark and bright with dark centre. With white light the central spot will be dark because the condition of darkness is independent of wavelength of light. This dark centre would be surrounded by few colour rings close to it and only the first few rings will be clear and after that there will be general illumination due to overlapping of different coloured rings.

(iv) *Newton's rings for lens in contact with a concave surface* If R_1 and R_2 are the radii of curvature of the convex surface of plano-convex lens L_1 and the concave surface of the concave glass plate L_2, then it can be easily shown that

$$\lambda = \frac{\mu(d_{n+p}^2 - d_n^2)}{4p}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad \text{...(4.62)}$$

where λ is the wave length of incident light and d_{n+p} and d_n are diameters of $(n+p)$th ring and nth ring. This relation is true for both dark and bright rings.

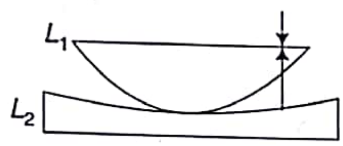


Fig. 4.22 Lens in contact with concave surface

(v) *Newton's rings for lens in contact with convex surface* If R_1 and R_2 are radii of curvature of the convex surface of plano-convex lens L_1 and convex surface of plano-convex lens L_2 then it can be easily shown that

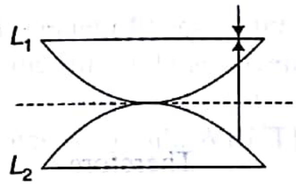


Fig. 4.23 Lens in contact with convex surface

$$\lambda = \frac{\mu(d_{n+p}^2 - d_n^2)}{4p}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \quad ...(4.63)$$

where λ is the wave length of incident light and d_{n+p} and d_n are diameters of $(n + p)$th ring and nth ring. This relation is true for both dark and bright rings.

(vi) *Calculation of fringe width* For two successive rings of diameters d_n and d_{n+1}, the fringe width β is given by,

$$\beta = \frac{d_{n+1} - d_n}{2}$$

Again, from Eq. (4.57) for $p = 1$

$$d_{n+1}^2 - d_n^2 = \frac{4\lambda R}{\mu}$$

or, $$(d_{n+1} - d_n)(d_{n+1} + d_n) = \frac{4\lambda R}{\mu}$$

or, $$d_{n+1} - d_n = \frac{4\lambda R}{\mu(d_{n+1} + d_n)} \simeq \frac{4\lambda R}{2d_n \mu} \quad [\text{Since } d_{n+1} + d_n \simeq 2d_n]$$

So, $$\beta = \frac{d_{n+1} - d_n}{2} = \frac{\lambda R}{\mu d_n} \quad ...(4.64)$$

Worked-out Examples

Example 4.1 The path difference between the two interfering rays at a point on the screen is 1/8th of a wavelength. Find the ratio of the intensity at this point to that at the centre of a bright fringe.

Sol. The intensity at any point is given by

$$I = a_1^2 + a_2^2 + 2a_1 a_2 \cos\delta$$

Here $a_1 = a_2 = a$ (say)

Therefore $I = 2a^2(1 + \cos\delta)$

At the centre, phase difference $\delta = 0$

$\therefore$ $I_0 = 2a^2(1 + \cos 0) = 2a^2(1 + 1) = 4a^2$

At the point, where the path difference is $\frac{\lambda}{8}$

$$\text{phase difference} = \frac{2\pi}{\lambda} \times \text{path difference}$$

$$= \frac{2\pi}{\lambda} \times \frac{\lambda}{8} = \frac{\pi}{4}$$

$$\therefore \qquad I_1 = 2a^2\left(1+\cos\frac{\pi}{4}\right) = 2a^2\left(1+\frac{1}{\sqrt{2}}\right)$$

Therefore $$\frac{I_1}{I_0} = \frac{2a^2\left(1+\frac{1}{\sqrt{2}}\right)}{4a^2} = \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right) = 0.85$$

Example 4.2 Two coherent sources whose intensity ratio is 100:1 produce interference fringes. Find the ratio of maximum intensity to the minimum intensity.

Sol. The ratio of maximum intensity to the minimum intensity is given by

$$\frac{I_{max}}{I_{min}} = \frac{(a_1+a_2)^2}{(a_1-a_2)^2} \qquad \text{...(1)}$$

Here $$\frac{I_1}{I_2} = \frac{a_1^2}{a_2^2} = \frac{100}{1}$$

or, $$\frac{a_1}{a_2} = \frac{10}{1} \quad \text{or,} \quad a_1 = 10\,a_2 \qquad \text{...(2)}$$

Substituting the value of a_1 from Eq. (2) in Eq. (1) we get,

$$\frac{I_{max}}{I_{min}} = \frac{(10\,a_2+a_2)^2}{(10\,a_2-a_2)^2} = \frac{121}{81}$$

Example 4.3 If the intensity at a point due to a light wave is 0.8 W/m^2, what would the intensity at that point when two such waves interfere at that point?

Sol. If they are in opposite phase, the resultant intensity (I_{min}) at the point of superposition would be equal to zero. But if they are in phase, then resultant intensity (I_{max}) at the point of superposition would be equal to

$$I_{max} = (a_1+a_1)^2 = 4a_1^2 = 4\times 0.8 = 3.2 \text{ W/m}^2$$

Example 4.4 In Young's double slit experiment, red light of 620 nm wavelength is used and the two slits are 0.3 mm apart. Interference fringes observed on a screen are 1.3 mm apart. Calculate the distance of the slits from the screen.

Sol. Fringe width $$\beta = \frac{D\lambda}{2d}$$

Here $\lambda = 620 \text{ nm} = 620\times 10^{-9}$ m

$2d$ = distance between the two slits

$= 0.3 \text{ mm} = 0.3\times 10^{-3}$ m

$$\therefore \qquad \beta = \frac{D\lambda}{2d}$$

or, $$D = \frac{\beta\times 2d}{\lambda} = \frac{1.3\times 10^{-3}\times 0.3\times 10^{-3}}{620\times 10^{-9}}$$

$$= 0.629 \text{ m}$$

Example 4.5 In an experiment using Young's double slits, the distance between the center of the interference pattern and tenth bright fringe on either side is 3.44 cm and the distance between the slits and the screen is 200 cm. If the wavelength of the light used is 5.89×10^{-5} cm, determine the separation between the slits.

[Question Bank, WBUT]

Sol. The distance of the nth bright fringe from the central maximum,

$$x_n = \frac{nD\lambda}{2d}$$

Here $n = 10, \quad D = 200$ cm

$\lambda = 5.89 \times 10^{-5}$ cm, $\quad x_n = 3.44$ cm

$$\therefore \quad 2d = \frac{nD\lambda}{x_n} = \frac{10 \times 200 \times 5.89 \times 10^{-5}}{3.44}$$

$$= 0.0342 \text{ cm}$$

So the diatance between the slits is 0.0342 cm.

Example 4.6 In a Young's double slit experiment the distance between the two coherent sources is 1.15 mm. Calculate the fringe width that would be observed on a screen place at a distance of 85 cm from the sources. The wavelength of light used is 5893 Å (WBUT 2005, 2008)

Sol. Fringe width $\quad \beta = \frac{D\lambda}{2d}$

Here $\lambda = 5893$ Å $= 5893 \times 10^{-8}$ cm

$2d = 1.15$ mm $= 0.115$ cm

$D = 85$ cm

$$\therefore \quad \beta = \frac{D\lambda}{2d} = \frac{85 \times 5893 \times 10^{-8}}{0.115}$$

$$= 0.43 \text{ cm}$$

Example 4.7 In a Young's double slit experiment, the fringes are formed at a distance of 1 m from the double-slit of separation 0.12 mm. Calculate the distance of 3rd dark band from the centre of the screen. Given wavelength of light $\lambda = 6000$ Å.

Sol. For $(n + 1)$th dark rings $\quad x_{n+1} = \left(n + \frac{1}{2}\right)\frac{D\lambda}{2d}$

For third dark fringe, $\quad n = 2$

Also, $\quad 2d = 0.012$ cm, $\quad D = 100$ cm, $\quad \lambda = 6 \times 10^{-5}$ cm

So, $$x_3 = \left(2 + \frac{1}{2}\right)\frac{100 \times 6 \times 10^{-5}}{0.012}$$

$$= 1.25 \text{ cm}$$

Example 4.8 In a Young's double slit experiment, the angular width of a fringe found on a distant screen is 0.1°. The wavelength of light used is 6000 Å. What is the spacing between the slits?

Sol. The angular fringe width (W_θ) is given by

$$W_\theta = \frac{\lambda}{2d}$$

∴ spacing between the slits ($2d$) is

$$2d = \frac{\lambda}{W_\theta}$$

Here $\lambda = 6000 \text{ Å} = 6000 \times 10^{-8} \text{ cm}$

$$W_\theta = 0.1° = \frac{0.1 \times \pi}{180} \text{ radian}$$

$$2d = \frac{6000 \times 10^{-8} \times 180}{0.1 \times \pi} = 3.44 \times 10^{-2} \text{ cm}$$

Example 4.9 A beam of light consisting of two wavelengths of 6500 Å and 5200 Å is used to obtain interference fringes in the Young's double slit experiment. What is the least distance from the central maximum when the bright fringes due to both the wavelengths coincide?

The distance between the slits is 2 mm and the distance between the plane of the slits and the screen is 120 cm.

Sol. The condition for least distance when the bright fringes due to two wavelengths to coincide is that the difference between their order of fringe width should be one.

$$x_n = \frac{Dn\lambda_1}{2d} = \frac{D(n+1)\lambda_2}{2d}$$

or, $$n\lambda_1 = (n+1)\,\lambda_2 \qquad \ldots(1)$$

Here $\lambda_1 = 6500 \text{ Å} = 6.5 \times 10^{-5} \text{ cm}$ and $\lambda_2 = 5200 \text{ Å} = 5.2 \times 10^{-5} \text{ cm}$

So from the Eq. (1)

$$6.5\,n = 5.2\,(n+1)$$

or, $$n = 4$$

Now from the equation

$$x_n = \frac{Dn\,\lambda_1}{2d}, \quad \text{we have,}$$

$$x_n = \frac{120 \times 4 \times 6.5 \times 10^{-5}}{0.2} = 0.156 \text{ cm.}$$

Example 4.10 In a double slit interference pattern at a point, the 10th order maximum is observed for light of wavelength 7000 Å. What order will be visible if the source of light is replaced by light of wavelength 5000 Å.

Sol. The condition of maxima in interference is given by

Path difference = $n\lambda$

Here $\lambda = 7000 \text{ Å}; n = 10$

So path difference $= 7000 \times 10^{-8} \times 10$ ...(1)

For light of wavelength 5000 Å

path difference $= n \times 5000 \times 10^{-8}$ ...(2)

Equating Eqs. (1) and (2) for path difference at the same point

$$n \times 5000 \times 10^{-8} = 10 \times 7000 \times 10^{-8}$$

$$n = \frac{10 \times 7000 \times 10^{-8}}{5000 \times 10^{-8}}$$

$$= 14$$

Hence, the 14th order of maximum will be observed.

Example 4.11 In a Newton's ring experiment, the diameter of 5th dark ring is 0.336 cm and the diameter of the 15th dark ring is 0.590 cm. Find the radius of the plano-convex lens if the wavelength of the light used is 5890 Å. [WBUT 2004]

Sol. Radius $R = \dfrac{D_{n+p}^2 - D_n^2}{4p\lambda} = \dfrac{D_{15}^2 - D_5^2}{4 \times 10 \times \lambda}$

Here $p = 10$, $D_5 = 0.336$ cm $D_{15} = 0.590$ cm

$$\lambda = 5890 \times 10^{-8} \text{ cm}$$

$$\therefore \quad R = \frac{(0.590)^2 - (0.336)^2}{4 \times 10 \times 5890 \times 10^{-8}}$$

$$= 99.83 \text{ cm}$$

Example 4.12 Light of wavelength 6000 Å falls normally on a thin wedge-shaped film of refractive index 1.4 forming fringes that are 2 mm apart. Find the angle of the wedge.

Sol. Given $\lambda = 6000 \times 10^{-10}$ m

$\mu = 1.4, \quad \beta = 2 \text{ mm} = 2.0 \times 10^{-3}$ m

The fringe width in the case of wedge-shaped film for normal incidence is given by

$$\beta = \frac{\lambda}{2\mu\theta}$$

or angle of wedge, $\theta = \dfrac{\lambda}{2\mu\beta}$

$$= \frac{6000 \times 10^{-10}}{2 \times 1.4 \times 2 \times 10^{-3}} = 1.07 \times 10^{-4} \text{ rad.}$$

Example 4.13 Newton's rings are observed in reflected light of wavelength 5900 Å. The diameter of the 10th dark ring is 0.5 cm. Find the radius of curvature of the lens and the thickness of the air film.

Sol. Here $\lambda = 5900 \times 10^{-10}$ m

$n = 10, \quad D_n = 0.5 \text{ cm} = 5 \times 10^{-3}$ m

(a) The diameter of the dark ring is given by

$$D_n^2 = 4\, n\lambda R$$

$$R = \frac{D_n^2}{4n\lambda} = \frac{(5 \times 10^{-3})^2}{4 \times 10 \times 5900 \times 10^{-10}} = 1.059 \text{ m}$$

The thickness of the air film is given by

$$2t = n\lambda$$

or,
$$t = \frac{n\lambda}{2} = \frac{10 \times 5900 \times 10^{-10}}{2} = 2.95 \times 10^{-6} \text{ m}$$

Example 4.14 A Newton's ring experiment is performed with a source of light having two wavelengths, $\lambda_1 = 6 \times 10^{-5}$ cm and $\lambda_2 = 4.5 \times 10^{-5}$ cm. It is found that the nth dark ring due to λ_1 coincides with $(n + 1)$ th dark ring due to λ_2. If the radius of curvature of the curved surface is 90 cm, find the diameter of the nth dark ring for λ_1.

Sol. The diameter of the nth ring is given by

$$D_n^2 = 4\, nR\lambda_1 \qquad \text{...(1)}$$

For $(n + 1)$th ring

$$D_{n+1}^2 = 4\,(n + 1)\, R\lambda_2 \qquad \text{...(2)}$$

The nth dark ring due to λ_1 coincides with $(n + 1)$th dark ring due to λ_2. Hence from Eqs. (1) and (2)

$$4nR\lambda_1 = 4\,(n + 1)\, R\lambda_2$$

or,
$$n\lambda_1 - n\lambda_2 = \lambda_2$$

or,
$$n(\lambda_1 - \lambda_2) = \lambda_2$$

$\therefore$
$$n = \frac{\lambda_2}{\lambda_1 - \lambda_2}$$

Here $\lambda_1 = 6 \times 10^{-5}$ cm and $\lambda_2 = 4.5 \times 10^{-5}$ cm; $R = 90$ cm

$\therefore$
$$n = \frac{4.5 \times 10^{-5}}{6 \times 10^{-5} - 4.5 \times 10^{-5}} = \frac{4.5}{1.5} = 3$$

So the diameter of the nth dark ring for λ_1 is

$$D = \sqrt{4 \times 3 \times 90 \times 6 \times 10^{-5}}$$
$$= 0.254 \text{ cm}$$

Example 4.15 A soap film of refractive index 1.33 is illuminated with light of different wavelengths at an angle of 45°. There is complete destructive interference for $\lambda = 5890$ Å. Find the thickness of the film.

Sol. Given
$$\mu = 1.33,\ i = 45°,\quad \mu = \frac{\sin i}{\sin r} \text{ or, } \sin r = \frac{\sin 45°}{1.33} = \frac{1}{\sqrt{2} \times 1.33} = 0.5317$$

$\therefore$
$$\cos r = \sqrt{1 - \sin^2 r}$$
$$= \sqrt{1 - (0.5317)^2} = 0.8469$$

For destructive interference

$$2\mu t \cos r = n\lambda$$

or,
$$2 \times 1.33 \times t \times 0.8469 = 1 \times 5890 \times 10^{-10}$$

$$\therefore \qquad t = \frac{5890 \times 10^{-10}}{2 \times 1.33 \times 0.8469} = 2.614 \times 10^{-7} \text{ m}$$

Example 4.16 In a Newton's ring experiment, the diameter of a dark ring is measured to be 2 mm. The medium in between plano-convex lens and glass plate is air. What would be the diameter of the same ring when the air film is replaced by a liquid of refractive index of 1.6?

Sol. The diameter of the dark ring is given by (for air)

$$D_n^2 = 4n\lambda R \qquad \text{...(1)}$$

For liquid, the diameter of the dark ring is

$$D_n'^2 = \frac{4n\lambda R}{\mu} \qquad \text{...(2)}$$

where μ is the refractive index of the liquid.

Now from Eqs. (1) and (2)

$$\frac{D_n^2}{D_n'^2} = \mu$$

or, $$D_n'^2 = \frac{D_n^2}{\mu}$$

or, $$D_n' = \sqrt{\frac{D_n^2}{\mu}} = \frac{D_n}{\sqrt{\mu}}$$

Here $D_n = 2 \text{ mm} = 0.2 \text{ cm}, \mu = 1.6$

$$\therefore \qquad D_n' = \frac{0.2}{\sqrt{1.6}} = 0.158 \text{ cm}$$

So the diameter of the ring is 0.158 cm.

Example 4.17 In a Newton's ring experiment, the diameter of the 25th bright ring is 7.5 mm and the radius of curvature of the plano-convex lens is 110 cm. Calculate the wave length of light used.

Sol. Given $n = 25$, $D_n = 7.5 \text{ mm} = 0.75 \text{ cm}$, $R = 110 \text{ cm}$, $\mu = 1$

The diameter of the nth bright ring is given by

$$D_n^2 = \frac{2(2n-1)\lambda R}{\mu}$$

or, $$\lambda = \frac{D_n^2 \mu}{2(2n-1)R} = \frac{(0.75)^2 \times 1}{2(2 \times 25 - 1) \times 110}$$

$$= 5217.99 \times 10^{-8} \text{ cm}$$

$$= 5218 \text{ Å}$$

Example 4.18 In a Newton's rings experiments the diameter of the 4th and 12th ring are 0.4 cm and 0.7 cm respectively. Find the diameter of the 20th dark ring.

Sol. Here $D_4 = 0.4$ cm $\quad D_{12} = 0.7$ cm

Now $$\lambda = \frac{D_m^2 - D_n^2}{4R(m-n)} = \frac{D_{12}^2 - D_4^2}{4 \times R \times 8} = \frac{(0.7)^2 - (0.4)^2}{4 \times 8 \times R}$$

or, $$\lambda R = \frac{0.49 - 0.16}{32} = 1.031 \times 10^{-2}$$

Now $$D_{20} = \sqrt{4R\,\lambda\, m} = \sqrt{4 \times 20 \times 1.031 \times 10^{-2}}$$
$$= 0.908 \text{ cm.}$$

Example 4.19 A film of oil of r.i 1.70 is placed between a plane glass plate and an equi-convex lens. The local length of the lens is 1 meter. Determine the radius of the 10th dark ring when light of wavelength 6000 Å is used. [WBUT 2008]

Sol. For equi-convex lens

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R} + \frac{1}{R}\right)$$ [Herer R is the radius of the lens. μ is refractive index of oil. f is the focal length of the lens]

$$\frac{1}{100} = (1.70 - 1)\frac{2}{R}$$
$$= 0.70 \times \frac{2}{R}$$

or, $$R = 70 \times 2 = 140 \text{ cm}$$

Now the radius of the 10th dark ring is $r_n = \sqrt{\frac{n\lambda R}{\mu}}$

$$r_{10} = \sqrt{\frac{10 \times 6000 \times 10^{-8} \times 140}{1.70}}$$

$$= \sqrt{\frac{8.4 \times 10^{-2}}{1.70}} = \frac{2.22}{10} = 0.22 \text{ cm}$$
$$= 2.2 \text{ mm}$$

Example 4.20 The coherence length for sodium light is 2.945×10^{-2} m. Calculate coherence time.

Sol. Coherence time $(\tau_c) = \frac{L}{c} = \frac{2.945 \times 10^{-2}}{3 \times 10^{8}}$
$$= 9.82 \times 10^{-11} \text{ s}$$

Review Exercises

Part 1: Multiple Choice Questions

1. Interference phenomena indicates
   (a) light is electromagnetic wave (b) rectilinear propagation of light
   (c) the wave nature of light (d) none of these
2. For a linear-shaped source of light, the nature of wave front is
   (a) plane (b) cylindrical (c) spherical (d) none of these

3. The nature of the wave front due to a point source of light is
(a) spherical (b) plane (c) cylindrical (d) none of these
4. For interference of light, the two sources should be
(a) spherical (b) non-coherent (c) coherent (d) cylindirical
5. Two sources will be coherent if they
(a) have a constant wavelength (b) have a constant phase difference
(c) have a constant amplitude (d) none of these
6. In interference, the law of conservation of energy is
(a) violated (b) not violated (c) partly violated (d) none of these
7. The relation between the path difference and phase difference is
(a) path difference $= \frac{\lambda}{2\pi} \times$ phase difference (b) path difference $= \frac{2\pi}{\lambda} \times$ phase difference
(c) path difference $= 2\pi \times$ phase difference (d) none of these
8. For constructive interference, the phase difference is an even multiple of
(a) $\frac{\pi}{2}$ (b) 2π (c) π (d) none of these
9. In Young's double slit experiment, coherent waves are produced by means of
(a) division of wave front (b) division of amplitude
(c) refraction (d) none of these
10. The fringes in Young's double slit experiment are
(a) equally spaced (b) not equally spaced
(c) partially equally spaced (d) none of these
11. Two waves having intensities in the ratio of 9:1 produce interference.
The ratio of maximum intensity to minimum intensity is equal to
(a) 4:1 (b) 4:9 (c) 10:8 (d) none of these
12. The fringe width of interference pattern of Young's double slit experiment is
(a) $\frac{D\lambda}{d}$ (b) $\frac{2d}{D\lambda}$ (c) $\frac{D\lambda}{2d}$ (d) $\frac{D}{\lambda d}$
[where λ is the wavelength of light, $2d$ is the distance between sources and D is the distance between source and screen]
13. The distance (x_n) of nth dark fringe from the central maximum is given by
(a) $\frac{D\lambda}{2d}$ (b) $\frac{2nD\lambda}{d}$ (c) $\frac{nD\lambda}{2d}$ (d) $(2n+1)\frac{D\lambda}{4d}$
14. In Newton's ring experiment, coherent waves are produced by means of
(a) division of wave front (b) diffraction
(c) division of amplitude (d) none of these
15. The centre of the Newton's rings for the reflected system of a monochromatic source of light is
(a) dark (b) bright (c) partially dark (d) none of these

16. The centre of the Newton's rings for the transmitted system of a monochromatic source of light is
    (a) dark (b) partially dark (c) bright (d) none of these
17. Newton's rings are formed
    (a) in a glass plate (b) in between plano-convex lens and glass plate
    (c) above the plano-convex lens (d) below the glass plate
18. The diameter of the Newton's ring is proportional to the
    (a) square of the radius of curvature of plano-convex lens
    (b) square root of the wavelength of the monochromatic light
    (c) wavelength of the monochromatic light
    (d) none of these
19. In Newton's ring experiment, rings are
    (a) of equal inclination and unequal thickness
    (b) of equal thickness and equal inclination
    (c) unequal inclination and unequal thickness
    (d) none of these
20. Radii of Newton's rings are proportional to
    (a) square root of natural numbers (b) square of the natural number
    (c) natural number (d) none of these
21. In Newton's rings experiment, the coherent sources are
    (a) spatial coherence (b) temporal coherence
    (c) partially spatial coherence (d) none of these
22. Stoke's law states that
    (a) the light wave reflecting from the surface of denser medium suffers a phase change of $\frac{\pi}{2}$
    (b) the light wave reflecting form the surface of denser medium suffers a phase change of π
    (c) the light wave reflecting form the surface of rearer medium suffers a phase change of $\frac{\pi}{2}$
    (d) none of these
23. If white light is used in Newton's ring experiment, then
    (a) a number of coloured rings will be observed
    (b) no rings will be observed
    (c) black and white rings will be observed
    (d) none of these
24. Two coherent sources of different intensities interfere with each other. The ratio of maximum intensity to the minimum intensity is 25. The intensities of the sources are in the ratio
    (a) 25:1 (b) 5:1 (c) 9:4 (d) 625:1
25. If interference takes place at some region, the light energy is
    (a) redistributed (b) created (c) destroyed (d) none of these

26. If Young's double slit experiment with one source of light and two slits be performed in water instead of air
 (a) the fringes will be smaller in number (b) the fringes will be narrower
 (c) the fringes will be broader (d) no fringes will be obtained
27. In Young's experiment with one source and two slits, if one slit is covered with black opaque paper,
 (a) the fringes will be narrower
 (b) the fringes will be darker
 (c) the fringes will be broader
 (d) no fringes will be obtained and the screen will have uniform illumination
28. A source of light emits light of frequencies between ν and $\nu + \Delta\nu$. If coherence time of the emergent light beam is τ_c, then
 (a) $\tau_c \propto \Delta\nu$ (b) $\tau_c \propto \frac{1}{\Delta\nu}$ (c) $\tau_c \propto \nu$ (d) $\tau_c \propto \frac{1}{\nu}$

Answers

1. (c)	2. (b)	3. (a)	4. (c)	5. (b)	6. (b)	7. (a)	8. (c)
9. (a)	10. (a)	11. (c)	12. (c)	13. (d)	14. (c)	15. (a)	16. (c)
17. (b)	18. (b)	19. (b)	20. (a)	21. (b)	22. (b)	23. (a)	24. (c)
25. (a)	26. (b)	27. (d)	28. (b)				

Short Questions with Answers

1. **Can we obtain interference pattern if two coherent sources are separated by distance less than the wavelength of light? Explain.**

 The fringe width $\beta = \frac{D\lambda}{2d}$. Now if $2d < \lambda$ then β will be sufficiently large which will result in uniform illumination. Hence, no interference fringe will be visible.

2. **Why are light waves from two different candles not seen to produce interference pattern?**

 The vibrations of a particle in the flame of the candle are so much influenced by inconsistent collisions of the surrounding particles that the phase of the vibration changes abruptly many time a second. Consequently, the waves generated from such vibrations are bound to have abrupt changes in phase too. In a flame there are numerous particles having all possible phases and for obvious reasons the waves coming from such vibrating particles are incapable of maintaining a constant phase relationship among them. So two similar candles or any two parts of the same candle cannot be regarded as coherent sources. Due to sudden and continual changes in the phase relation of these waves, the position of maxima and minima in the fringe system undergo very rapid charge, with the result that the screen is seen informally illuminated.

3. **If the amplitudes of the two interfering waves are different, find out the resultant amplitude and intensity?**

 When two beams of light of amplitude a_1 and a_2 from two sources having same frequency fall upon a screen, then the resultant amplitude (A) will be given by

$$A^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos\delta$$

and the resultant intensity will be given by

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta$$

where δ is the phase difference between the two beams and I_1 and I_2 are the intensities due to two beams.

4. **What is the effect on the fringe pattern if we do not make the simplification $S_2P + S_1P \simeq 2D$**

   In this case we have to use the exact expression of the path difference

$$S_2P - S_1P = \sqrt{D^2 + (x+d)^2} - \sqrt{D^2 + (x-d)^2}$$

   When the path difference equal to $n\lambda$ we get maximum. Obviously, the fringes will not be equally spaced.

5. **What happens to fringe width when the double slit set up is immersed in water?**

   We know fringe width $\beta = \dfrac{\lambda D}{2d}$. Since λ decreases in water so fringe width will decreased.

6. **What would be the nature of interference pattern produced by wide slit?**

   Wide slit can be treated as a combination of large number of narrow slits placed side by side. Then each of these narrow slits produce own interference pattern that overlaps with interference pattern of others. As a result, we have natural illumination without variation in intensity.

7. **A double slit is first illuminated by violet light and then by red light. Which colour shows wider interference pattern beyond the double slit?**

   The fringe width is directly proportional to the wavelength. Since $\lambda_r > \lambda_v$, so red light produces wider interference pattern.

8. **Do interference phenomena violate the principle of conservation of energy?**

   See Article 4.8

9. **What is the difference between temporal coherence and spatial coherence?**

   The temporal coherence of any wavefield implies the possibility of predicting phase and amplitude at a point in space at different instants of time. A monochromatic wavefield is temporally coherent. The spatial coherence is concerned with phase correlation between two wavefields at two space points at the same instant of time.

10. **In Young's double slit experiment, fringes appear as straight line. Explain.**

    Actually, in Young's double slit experiment, the region in which constructive interference and destructive interference take place form hyperboloid. But intersections of these hyperboloid and the screen forms straight lines. Therefore, fringes come into sight as straight line on the screen.

11. **How does the interference pattern by reflection in thin film differ from that of refraction (transmission)?**

    In the reflected system there is an additional path difference of $\dfrac{\lambda}{2}$ between the two rays producing interference. Here, one ray suffers reflection at a denser medium, while in the transmitted system it is not so.

    Thus, for the same path difference in the reflected system a bright band will correspond to a dark band in transmitted system and vice versa. Thus, the two systems are complementary.

12. Explain why an excessively thin film appears black in reflected light.

The effective path difference between the interfering reflected rays is $2\mu t \cos r - \frac{\lambda}{2}$. When the film is extremely thin, so that t is practically zero, the effective path difference is $\frac{\lambda}{2}$. This is the condition for minimum intensity. Hence the film appears dark.

13. Why are Newton's rings formed?

Newton's rings are formed due to interference between the light waves reflected from the top and the bottom surfaces of the air film formed between the lens and the glass plate.

14. What change will take place in Newton's ring system if a liquid film is used instead of air film.

let r_a be the radius of the nth bright-ring with air-film and r_l is the radius of the same ring with liquid film, then

$$r_a^2 = 4\,R(2n+1)\frac{\lambda}{2}, \quad r_l = \frac{4R}{\mu}(2n+1)\frac{\lambda}{2}$$

$$\therefore \quad r_l^2 = \frac{r_a^2}{\mu} \quad \text{as} \quad \mu > 1 \text{ and } r_l < r_a$$

Part 2: Descriptive Questions

1. (i) What are coherent sources?
   (ii) State the condition to be fulfilled for the production of sustained interference fringes. [WBUT 2004]
   (iii) Is light energy destroyed in the region of destructive interference?
   (iv) Why are light waves from two different candles not seen to interfere?
2. What is meant by interference of light?
   Show that the dark and bright fringes produced in Young's experiment are equally spaced.
3. (i) In an interference experiment d is the distance between the two coherent sources of light with wavelength λ and D is the source-screen distance. Show that the separation between the two consecutive dark bands is given by $\beta = \frac{D\lambda}{d}$ [WBUT 2004]
   (ii) Interference phenomenon does not violet the principle of conservation of energy. Justify it. [WBUT 2008]
4. (i) What are coherent sources?
   (ii) Show that the law of conservation of energy is not violated in the interference process.
   (iii) Will the radius of the Newton's ring change if air is replaced by oil between the lens and the glass plate? [WBUT 2005]
5. (i) Explain briefly with necessary theory how Newton's rings are formed [WBUT 2003]
   (ii) Why is the centre of Newton's rings dark? [WBUT 2004]
   (iii) Would you observe Newton's ring with transmitted light?
6. (i) What are coherent and non-coherent sources of light?

(ii) Show that the expression of the radius of curvature of the convex lens used in Newton's ring apparatus is $R = \dfrac{D_{m+n}^2 - D_m^2}{4n\lambda}$.

(iii) What are basic requirements for obtaining permanent interference pattern? [WBUT 2003]

7. (i) Explain the colour phenomenon exhibited by a thin film.
   (ii) Explain with necessary theory how you can determine the refractive index of a liquid by means of Newton's ring.
8. (i) Established the conditions of constructive and destructive interference of light coming out from two coherent sources.
   (ii) Show how the energy is conserved in interference phenomena.
   (iii) If the distance between the two coherent sources of light with wavelength λ is d and D is the source screen distance then show that fringe width separation $x = \dfrac{D\lambda}{d}$. [WBUT 2002]
9. (i) Define temporal coherence and spatial coherence?
   (ii) Explain the difference between temporal and spatial coherence.
10. (i) A double silt is illuminated first by red light and then by violet light. Which colour gives the wider interference pattern beyond the double slit?
    (ii) Why is laser source more monochromatic than conventional source?
11. Define lateral coherence width. Explain the term that degree of contrast of fringe pattern is a measure of degree of spatial coherence and monochromaticity is a measure of temporal coherence.

Part 3: Numerical Problems

1. Two coherent sources having intensity in the ratio 81:1 produce interference fringes. Find the ratio of the maximum to minimum intensity of fringe system **[25:16]**
2. In Young's double slit experiment with a light of wavelength 589.3 nm separation between the slits is 1 mm. Find the fringe width on a screen 1 m away. **[0.05893 cm]**
3. Two coherent sources are placed 0.2 mm apart and the fringes are observed on a screen 1 m away. It is found that with a certain monochromatic source of light, the fourth bright fringe is situated at a distance of 10.0 mm from the central fringe. Find the wavelength of light. **[5000 Å]**
4. A double-slit arrangement produces interference fringes for sodium light ($\lambda = 5800$ Å) that are 0.20° apart. For what wavelength would the angular separation be 10 per cent greater? **[6479 Å]**
5. Newton's rings are formed by light reflected normally from a plano-convex lens and a plane glass plate with a liquid between them. The diameter of the nth and $(n + 10)$th bright rings are 2.18 mm and 4.51 mm respectively. Calculate the refractive index of the liquid. Radius of curvature of the lens is 90 cm and wave length of light employed is 589.3 nm. **[1.7]**
6. If the diameters of two consecutive Newton's rings in reflected light of wavelength 5890 Å are 2.0 and 2.02 cm respectively, what is the radius of the lens surface in contact with plane glass surface? **[341.2 cm]**
7. In a Newton's rings experiment, the diameter of 5th dark ring is reduced to half of its value after introducing a liquid below the convex surface and glass plate. Calculate the refractive index of the liquid. **[1.4]**

8. Newton's rings are formed with a source of light containing two wavelengths λ_1 and λ_2. If mth order dark ring due to λ_1 coincides with the $(m + 1)$th order dark ring due to λ_2 prove that the radius of mth dark ring of λ_1 is equal to $\sqrt{\dfrac{\lambda_1 \lambda_2 R}{\lambda_1 - \lambda_2}}$, where R is the radius of curvature of the lower curved surface.

9. In a Young's double slit experiment the distance between the slits is 0.1 mm and the perpendicular distance of the screen from the plane of the slits is 50 cm. Find the separation on the screen between maxima for violet light ($\lambda = 400$ mm) and the red light ($\lambda = 700$ mm) in the first order.

$\left[\textbf{Hints: } x_n = \dfrac{n\lambda D}{2d};\quad x_1' - x_1 = \dfrac{D}{2d}(\lambda_2 - \lambda_1) = 1.5 \text{ mm}\right]$

CHAPTER

5

Diffraction of Light

5.1 INTRODUCTION

The bending of light around the corner of an obstacle and spreading of light wave into the geometrical shadow of that obstacle is called diffraction of light. Fresnel gave a satisfactory explanation of this on the basis of wave theory. According to him, the wavelength of light is so small that we cannot see the bending of light ray round the corners; so the rectilinear propagation of light is only an approximation. Diffraction also occurs in sound and because of this, we can listen to a speaker when we stand behind a pillar, i.e., even when we are not in the direct line with the speaker.

In order to explain the phenomenon of diffraction of light, one can represent a source of smallest possible size (the so-called point source) by a point. Let us represent a point source by S which is in front of a screen MN [Fig. (5.1a)]. A narrow opaque obstacle AB is placed in between the source and the screen. If one assumes rectilinear propagation of light, then AB will cast its shadow on the screen MN which has been shown by CD. But, one can observe that dark bright fringes near the edge are present. Similarly, if the light from the source S is allowed to pass through an aperture and falls on a screen MN [Fig. 5.1(b)], then it is seen that the intensity of the light does not fall exactly to zero outside the area CD, instead some fluctuations are observed in the form of dark and bright fringes. Therefore, from the above observations it is evident that the light bends to an extent while passing through a small aperture or by the edge of a small obstacle.

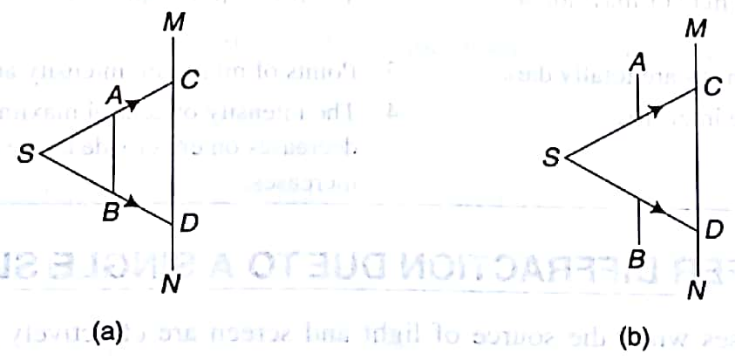


Fig. 5.1(a) Narrow opaque obstacle AB in between the source and the screen

Fig. 5.1(b) Small aperture AB in between the source and the screen

Definition of diffraction The phenomenon of bending of light round the sharp corners and spreading into the region of the geometrical shadow is called diffraction.

The light waves are diffracted only when the size of the obstacle is comparable to the wavelength of light.

The effects of diffraction increase with the increase of wavelength. Indeed since the light wavelengths are very small (~ 5×10^{-7} m), the effects due to diffraction are not readily observed.

5.2 DIFFERENT TYPES OF DIFFRACTION PHENOMENA

There are two types of diffraction phenomena known as ***Fresnel diffraction*** and ***Fraunhofer diffraction***. Some important characteristics of these categories of diffraction is as under:

(i) In Fresnel's diffraction, either the source or screen or both are at finite distances from the obstacle and thus the distances are important.

In Fraunhofer's diffraction, the source of light and the screen are at infinite distances from the obstacle. In this class, the inclination is important.

(ii) In Fresnel's diffraction, the incident wave front is either spherical or cylindrical while in Fraunhofer's diffraction, the incident wave front is generally plane.

(iii) In Fresnel's diffraction, the centre of diffraction pattern may be bright or dark depending upon the number of Fresnel's zone, while in Fraunhofer's diffraction the centre of the diffraction pattern is always bright.

(iv) In real life, Fresnel's diffraction is more common. It is however difficult to treat and understand. Fraunhofer's diffraction is a limiting case of Fresnel's diffraction and it is easier to handle mathematically.

5.3 DIFFERENCE BETWEEN INTERFERENCE AND DIFFRACTION

Both of interference and diffraction are the resultant of superposition of waves. Both these effects may take place simultaneously. The difference between the two phenomena are given in the following table.

Interference	*Diffraction*
1. The interference occurs between two separate wave fronts originating from two coherent sources.	1. It is the interference that occurs between the secondary wave-lets originating from the exposed part of the same wave front.
2. The interference fringes may or may not be equally spaced.	2. The diffraction fringes are never equally spaced.
3. Points of minimum intensity are totally dark.	3. Points of minimum intensity are not totally dark.
4. The maxima are of same intensity.	4. The intensity of central maximum is maximum and decreases on either side as the order of maximum increases.

5.4 FRAUNHOFER DIFFRACTION DUE TO A SINGLE SLIT

Fraunhofer diffraction arises when the source of light and screen are effectively at infinite distance from diffracting aperture.

(a) *Single slit* A slit is a rectangular aperture whose length is large as compared to its breadth. The width of the slit should be at least comparable with the wavelength of the light used but never less than the wavelength.

(b) *Principle* The study of diffraction pattern is based on the super-position of Huygens' secondary wavelet which are supposed to be generated at every point on the wavefront when it occupies the plane of the slit.

AB is a long narrow slit of width 'a' placed perpendicular to the plane of the paper and illuminated by a parallel beam of monochromatic light of wavelength λ. The figure shows a cross-section of the slit and O is the centre of the slit. Let i is the angle of incidence with the normal to the plane of the slit. Due to the diffraction these rays will generate secondary wavelets in all possible directions and are focussed on the focal plane of the converging lens L. Here parallel diffracted rays are focussed by the lens on the screen at P to form the image of the slit. So the point of observation P is also at infinity. Obviously it is a case of Fraunhofer class of diffraction.

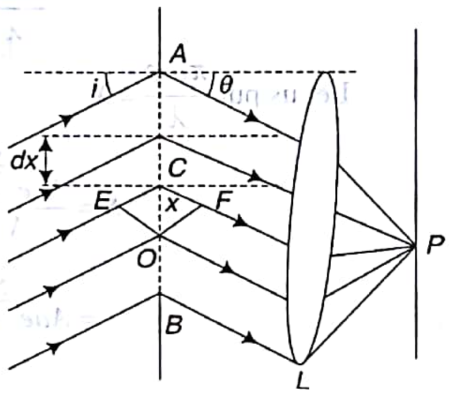


Fig. 5.2 Fraunhofer diffraction through single slit.

(c) *Mathematical Study of Intensity* To study the diffraction phenomenon, let us consider a diffracting element of width dx at C at a distance x from O.

Through the slit let us draw incident and diffracted rays as shown in the Fig. (5.2). From O draw two perpendicular OE and OF on the incident and diffracted rays. Then OE and OF represent incident and diffracted wave front. We shall find out the intensity of light at P.

Let the equation of vibration on the incident wave front OE be represented by $y = Ae^{i\frac{2\pi}{\lambda}ct}$ where c is the velocity of light. So, the equation of vibration on the diffracted wave front OF is given by

$$y = Ae^{i\frac{2\pi}{\lambda}(ct - ECF)} \quad \ldots(5.1)$$

But
$$ECE = EC + CF = x \sin i + x \sin \theta$$
$$= x(\sin i + \sin \theta)$$
$$= x\varphi \text{ (say)}$$

So the Eq. (5.1) reduces to

$$y = Ae^{i\frac{2\pi}{\lambda}(ct - x\varphi)} \quad \ldots(5.2)$$

Now neglecting the effects of disturbance due to inclination, the disturbance at P due to element dx at C is given by,

$$ds = ydx = Ae^{i\frac{2\pi}{\lambda}(ct - x\varphi)}dx \quad \ldots(5.3)$$

Hence the total disturbance at P due to entire slit is given by,

$$S = \int ds = \int_{-a/2}^{a/2} Ae^{i\frac{2\pi}{\lambda}(ct - x\varphi)}dx \quad \ldots(5.4)$$

$$= Ae^{i\frac{2\pi}{\lambda}ct}\int_{-a/2}^{a/2} e^{-i\frac{2\pi}{\lambda}x\varphi}dx$$

$$= \frac{Ae^{i\frac{2\pi}{\lambda}ct}}{\frac{\pi\varphi}{\lambda}}\left[\frac{e^{+i\frac{\pi a\varphi}{\lambda}} - e^{-i\frac{\pi a\varphi}{\lambda}}}{2i}\right]$$

$$= \frac{Aae^{i\frac{2\pi}{\lambda}ct}}{\frac{\pi a\varphi}{\lambda}}\left[\frac{e^{+i\frac{\pi a\varphi}{\lambda}} - e^{-i\frac{\pi a\varphi}{\lambda}}}{2i}\right]$$

Let us put $\frac{\pi a\varphi}{\lambda} = X$

$$S = \frac{Aae^{i\frac{2\pi}{\lambda}ct}}{X}\frac{e^{iX} - e^{-iX}}{2i}$$

$$= Aae^{i\frac{2\pi}{\lambda}ct}\left[\frac{\sin X}{X}\right]$$

$$= \left(Aa\frac{\sin X}{X}\right)e^{i\frac{2\pi}{\lambda}ct} \qquad \ldots(5.5)$$

Hence the vibration is simple harmonic. The intensity at P is proportional to the square of amplitude. Let the constant of proportionality is unity. The intensity at P is given by,

$$I = A^2 a^2 \frac{\sin^2 X}{X^2} = A^2 a^2 \frac{\sin^2 \frac{\pi a\varphi}{\lambda}}{\left(\frac{\pi a\varphi}{\lambda}\right)^2}$$

$$= A^2 a^2 \frac{\sin^2\left\{\frac{\pi a}{\lambda}(\sin i + \sin\theta)\right\}}{\left\{\frac{\pi a}{\lambda}(\sin i + \sin\theta)\right\}^2} \qquad \ldots(5.6)$$

(d) Conditions for Maxima and Minima

Sine we have

$$I = \frac{A^2a^2 \sin^2 X}{X^2}$$

$$\therefore \qquad \frac{dI}{dX} = A^2a^2\left[\frac{X^2\, 2\sin X\cos X - \sin^2 X\cdot 2X}{X^4}\right]$$

$$= A^2a^2\left[\frac{2X\sin X\cos X - 2\sin^2 X}{X^3}\right] \qquad \ldots(5.7)$$

For the above Eq. (5.7), we shall get the condition of maxima and minima by putting $\frac{dI}{dX} = 0$

i.e. $$\frac{d}{dX}\left[A^2a^2\frac{\sin^2 X}{X^2}\right] = 0$$

or, $$A^2 a^2 2 \sin X \frac{[X \cos X - \sin X]}{X^3} = 0$$

We have as conditions

(a) $X = \infty$ (b) $\sin X = 0$ (c) $X \cos X - \sin X = 0$

For condition (a) we have $X = \dfrac{\pi a \varphi}{\lambda} = \infty$

This will be true, when $\lambda = 0$. As $\lambda \neq 0$, so this condition is invalid.

From condition (b) $\sin X = 0$

or, $$\sin \frac{\pi a \varphi}{\lambda} = 0$$

i.e., $$\frac{\pi a \varphi}{\lambda} = n\pi \quad \text{when } n \text{ is an integer}$$

Hence $$\varphi = \frac{n\lambda}{a} \qquad \ldots(5.8)$$

If we find $\dfrac{d^2 I}{dX^2}$ and put the condition in Eq. (5.8) in it then $\dfrac{d^2 I}{dX^2}$ = + ve. So the Eq. (5.8) gives the condition for minimum and the other successive minima are given by

$$\varphi_1 = \pm \frac{\lambda}{a}, \quad \varphi_2 = \pm \frac{2\lambda}{a}, \quad \varphi_3 = \pm \frac{3\lambda}{a} \ldots$$

From the above values of φ we can conclude that the minima are equidistant.

From condition (c) $\tan X = X$, this gives the condition for maxima. Now the value of X satisfying the above equation is obtained by drawing two curves (i) $Y = X$ and $Y = \tan X$ on the same scale. Then the point of intersection of the two curves [Fig. 5.3(b)] gives the values of X, satisfy the equation $\tan X = X$, the maximum value of intensity occur at

(i) $X_0 = 0$ giving $I_0 = \underset{X_0 \to 0}{\text{Lt}} \dfrac{A^2 a^2 \sin^2 X}{X^2} = A^2 a^2$ (central band)

(ii) $X_1 = \pm 1.43\,\pi$ giving $I_1 = 0.0469\, A^2 a^2$ and

$$\varphi_1 = \pm 1.43 \frac{\lambda}{a}$$

(iii) $X_2 = \pm 2.46\,\pi$ giving $I_2 = 0.0168\, A^2 a^2$ and

$$\varphi_2 = \pm 2.46 \frac{\lambda}{a}$$

(iv) $X_3 = \pm 3.47\,\pi$ giving $I_3 = 0.0083\, A^2 a^2$ and

$$\varphi_3 = \pm 3.47 \frac{\lambda}{a}$$

Thus, it is obvious from the values of I_0, I_1, I_2 etc., that the diffraction pattern consists of a bright central maximum followed by minimum of zero intensity and then secondary maxima of decreasing intensity on either side of it [Fig. 5.3(a)] and the maxima are not equidistant but at higher order these are found to be equidistant.

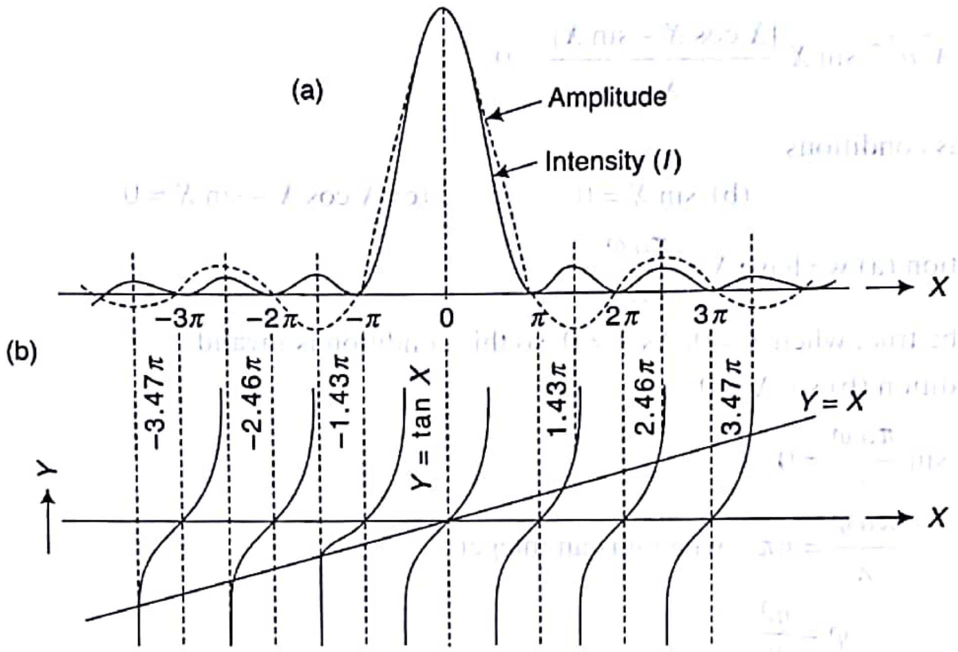


Fig. 5.3 (a) Intensity (solid line)/Amplitude (dotted line) distributions due to single slit **(b) Graphical method for determining the roots of the equation tan $X = X$.**

The angle of diffraction θ_1 for the 1st minimum on either side of the central maximum is given by Eq. (5.8)

$$a \sin \theta_1 = \lambda \qquad \left[\because \varphi = \frac{n\lambda}{a},\ \sin i + \sin \theta = \frac{n\lambda}{a}.\ \text{For normal incidence } i = 0 \text{ and for first minimum } n = 1 \text{ so } \sin \theta_1 = \frac{\lambda}{a}\right]$$

or, $$\theta_1 = \frac{\lambda}{a}$$

Therefore, the angular width of principal maximum ($2\theta_1$) is inversely proportional to the width (a) of the slit.

5.5 FRAUNHOFER DIFFRACTION DUE TO A DOUBLE SLIT

The diffraction pattern due to double slit consists of diffraction fringes caused by rays diffracted from two slits superimposed on the interference fringes caused by rays coming from each pair of corresponding points on the two slits.

(a) Double slit Double slit is an arrangement where two single slits are parallelly placed on the same plane. The width of each slit are generally identical and much smaller than their lengths.

The slits have an opaque space between them and width of both the slits and opaque space should be of the order of wavelength used or greater but never less than the wavelength of the light.

Adjoining diagram [Fig. (5.4)] represents the section of the slit. Here MN and RS are two slits on which beams of parallel rays are incident. And a is the width of each slit and b is the width of opaque space. O and O_1 are midpoint of MN and RS respectively and d is the distance between O and O_1 which is equal to $(a + b)$. $S_1 S_2$, the plane of the slits is perpendicular to the plane of the paper. And i is the angle of incidence with the normal to the plane of the slits. The rays are diffracted in all possible directions. Let us consider a beam of

rays for which the angle of diffraction is θ. L is the converging lens which focuses the diffracted rays on its focal plane.

As the source and the point of observation are effectively at infinity, obiviously it is a case of Fraunhofer class of diffraction.

(b) Mathematical/Analytical study of Intensity To find the intensity on the screen let us consider a diffracting element of width dx at P where $PO_1 = x$. From O_1 let us draw two perpendiculars O_1A and O_1B on the incident and diffracted rays.

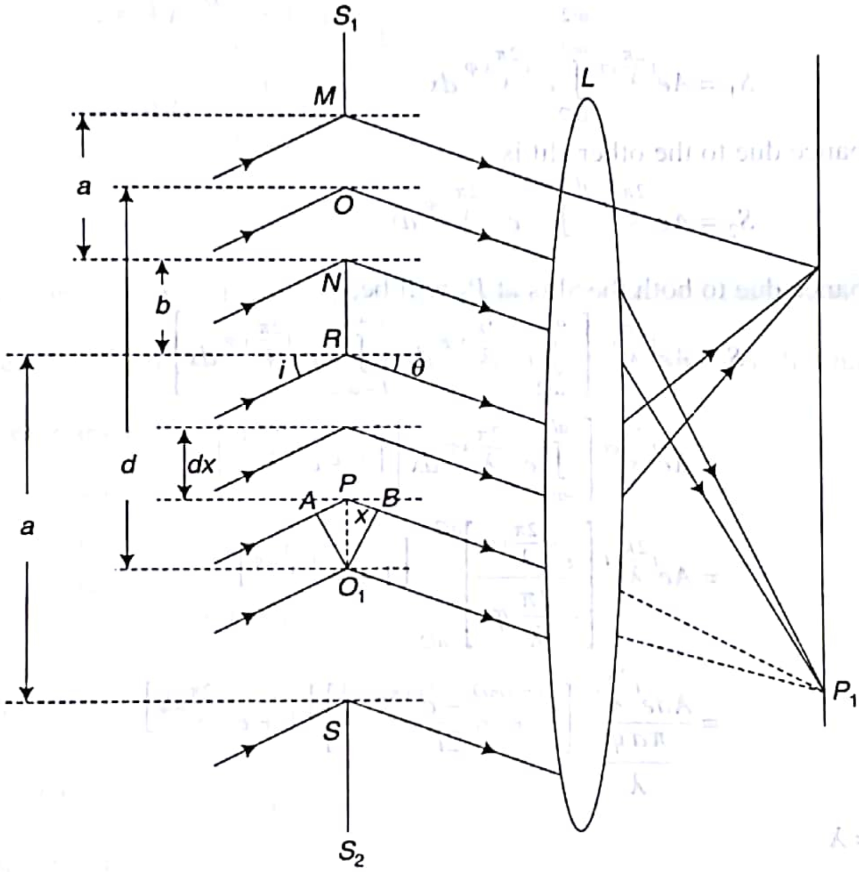


Fig. 5.4 Fraunhofer diffraction through double slit.

Let the equation of vibration of the incident wave front O_1A is represented by $y = Ae^{i\frac{2\pi}{\lambda}ct}$ where c is the velocity of light in free space, A is the amplitude of the wave and λ is the wavelength.

The equation of vibration of diffracted wave front O_1B is given by

$$y = Ae^{i\frac{2\pi}{\lambda}(ct - APB)} \qquad \ldots(5.9)$$

But
$$APB = AP + PB = x\sin i + x\sin\theta$$
$$= x(\sin i + \sin\theta)$$
$$= x\varphi \qquad [\varphi = \sin i + \sin\theta] \qquad \ldots(5.10)$$

Equation (5.9) reduces to $y = Ae^{i\frac{2\pi}{\lambda}(ct - x\varphi)}$

Now, the disturbance at P_1 due to the element dx at P is given by

$$ds = y\,dx = Ae^{i\frac{2\pi}{\lambda}(ct - x\varphi)}\,dx$$

Hence the total disturbance at P due to entire slit RS is given by

$$\int ds = \int_{-a/2}^{a/2} Ae^{i\frac{2\pi}{\lambda}(ct - x\varphi)}\,dx \qquad \text{...(5.11)}$$

$$= Ae^{i\frac{2\pi}{\lambda}ct}\int_{-a/2}^{a/2} e^{-i\frac{2\pi}{\lambda}x\varphi}\,dx$$

or,

$$S_1 = Ae^{i\frac{2\pi}{\lambda}ct}\int_{-a/2}^{a/2} e^{-i\frac{2\pi}{\lambda}x\varphi}\,dx$$

Hence the disturbance due to the other slit is

$$S_2 = Ae^{i\frac{2\pi}{\lambda}ct}\int_{d-a/2}^{d+a/2} e^{-i\frac{2\pi}{\lambda}x\varphi}\,dx$$

Hence the disturbance due to both the slits at P_1 will be,

$$S = Ae^{i\frac{2\pi}{\lambda}ct}\left[\int_{-a/2}^{a/2} e^{-i\frac{2\pi}{\lambda}x\varphi}\,dx + \int_{d-a/2}^{d+a/2} e^{-i\frac{2\pi}{\lambda}x\varphi}\,dx\right]$$

$$= Ae^{i\frac{2\pi}{\lambda}ct}\left[\int_{-a/2}^{a/2} e^{-i\frac{2\pi}{\lambda}x\varphi}\,dx\right]\left[1 + e^{-i\frac{2\pi}{\lambda}d\varphi}\right]$$

$$= Ae^{i\frac{2\pi}{\lambda}ct}\left[\frac{e^{-i\frac{2\pi}{\lambda}x\varphi}}{-\frac{2\pi}{\lambda}\varphi i}\right]_{-a/2}^{a/2}\left[1 + e^{-i\frac{2\pi}{\lambda}d\varphi}\right]$$

$$= \frac{Aae^{i\frac{2\pi}{\lambda}ct}}{\frac{\pi a\varphi}{\lambda}}\left[\frac{e^{i\pi a\varphi/\lambda} - e^{-i\pi a\varphi/\lambda}}{2i}\right]\left[1 + e^{-i\frac{2\pi}{\lambda}d\varphi}\right]$$

Let us put $\dfrac{\pi a\varphi}{\lambda} = X$

$$\therefore \qquad S = \frac{Aae^{i\frac{2\pi}{\lambda}ct}}{X}\left[\frac{e^{iX} - e^{-iX}}{2i}\right]\left[1 + e^{-i\frac{2\pi}{\lambda}d\varphi}\right]$$

$$= Aae^{i\frac{2\pi}{\lambda}ct}\left[\frac{\sin X}{X}\right]\left[1 + \cos\frac{2\pi}{\lambda}d\varphi - i\sin\frac{2\pi}{\lambda}d\varphi\right]$$

Here we see that the effective amplitude of vibration at P_1 due to two slits is

$$Aa\left(\frac{\sin X}{X}\right)\left(1 + \cos\frac{2\pi}{\lambda}d\varphi - i\sin\frac{2\pi}{\lambda}d\varphi\right)$$

Now the resultant intensity at P_1 is proportional to the square of the amplitude. Let the constant of proportionality be unity.

$\therefore$ intensity at P_1 is given

$$I = A^2a^2\frac{\sin^2 X}{X^2}\left[\left(1 + \cos\frac{2\pi}{\lambda}d\varphi\right)^2 + \sin^2\frac{2\pi}{\lambda}d\varphi\right]$$

$$= 2A^2a^2 \frac{\sin^2 X}{X^2}\left(1 + \cos\frac{2\pi}{\lambda} d\varphi\right)$$

$$= 2A^2a^2 \frac{\sin^2 X}{X^2} \cdot 2\cos^2\frac{\pi d\varphi}{\lambda}$$

$$\therefore \quad I = 4a^2A^2 \frac{\sin^2 X}{X^2}\cos^2\frac{\pi d\varphi}{\lambda}$$

$$I = 4I_0 \frac{\sin^2 X}{X^2}\cos^2\frac{\pi d\varphi}{\lambda} \qquad \ldots(5.12)$$

$$I = 4I_0 \frac{\sin^2 X}{X^2}\cos^2 Y \qquad \ldots(5.13)$$

where $Y = \frac{\pi d\varphi}{\lambda}$

The factor $\frac{\sin^2 X}{X^2}$ is similar to that for the single slit pattern. The second factor $\cos^2\frac{\pi d\varphi}{\lambda}$ is the characteristic of the interference pattern due to the disturbance coming from the two slits having a phase difference $\frac{2\pi}{\lambda} d\varphi$. The minimum intensity of interference

$$\cos^2\frac{\pi d\varphi}{\lambda} = 0$$

or, $$\frac{\pi d\varphi}{\lambda} = (2n+1)\frac{\pi}{2},$$

or, $$\varphi(a+b) = (2n+1)\frac{\lambda}{2} \quad \begin{bmatrix} d = a+b \\ \varphi = \sin i + \sin\theta \end{bmatrix}$$

or, $$(\sin i + \sin\theta)(a+b) = (2n+1)\frac{\lambda}{2}$$

For normal incidence, $i = 0$

Thus $$(a+b)\sin\theta = (2n+1)\frac{\lambda}{2}$$

or, $$\sin\theta = \frac{2n+1}{(a+b)}\frac{\lambda}{2}$$

So we shall get successive minima as

$$\frac{\lambda}{2(a+b)}, \quad \frac{3\lambda}{2(a+b)}, \quad \frac{5\lambda}{2(a+b)}, \quad \ldots$$

From this it is clear that minima are equidistant. The intensity of the interference pattern will be maximum

when $$\cos^2\frac{\pi d\varphi}{\lambda} = \pm 1$$

or, $$\frac{\pi d\varphi}{\lambda} = n\pi$$

or, $\quad (a+b)\sin\theta = n\lambda$, $\quad$ where $n = 0, \pm 1, \pm 2 \ldots$ etc. $\quad$ [for normal incidence $i = 0$ and $d = a + b$]

or, $$\sin\theta = \frac{n\lambda}{a+b}$$

Hence the successive maxima are at [taking +ve sign]

$$\frac{\lambda}{a+b}, \quad \frac{2\lambda}{a+b}, \quad \frac{3\lambda}{a+b}, \quad \ldots$$

Thus we see that maxima are equidistant. The interference term $\cos^2\frac{\pi d\varphi}{\lambda}$ gives a set of equidistant dark and bright fringes as in Young's double slit interference experiment [Fig. 5.5(a)]. The diffraction term $\frac{\sin^2 X}{X^2}$ gives a central maximum at $X = 0$ i.e., in the direction $\theta = 0$. The maxima also occur at values of X approaching $\pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$ etc. These are secondary diffraction maxima.

The minima are obtained when $\frac{\sin^2 X}{X^2} = 0$ or $\sin X = 0$ but $X \neq 0$

$\therefore$ $$X = m\pi$$

or, $$\frac{\pi a \sin\theta}{\lambda} = m\pi \quad \text{[For normal incidence } i = 0\text{]}$$

or, $$a\sin\theta = m\lambda, \quad m = \pm 1, \pm 2 \ldots \text{ except } 0$$

These minima are known as diffraction minima. So based on diffraction term, the pattern consists of a central maximum in the direction $\theta = 0$ with alternate minima and secondary maxima of diminishing intensity on either side [Fig. 5.5(b)]. The intensity distribution in the resultant diffraction pattern is shown in Fig. 5.5(c).

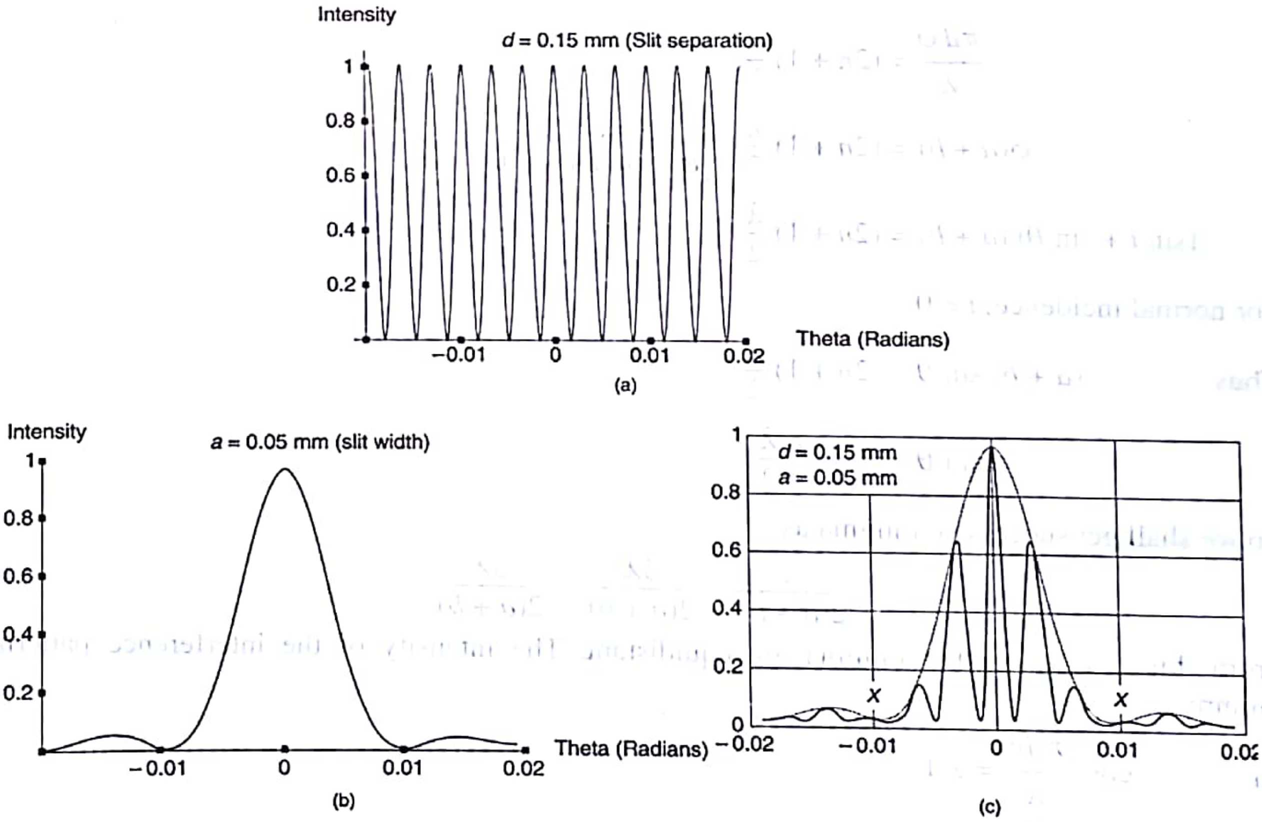


Fig. 5.5 (a) Double slit interference pattern, (b) Single slit diffraction pattern. (c) Intensity distribution of the resultant diffractions pattern

Figure 5.5 shows graphs of intensity versus angular position for (a) ideal interference pattern for a double slit (b) the diffraction pattern for single slit, and (c) resultant diffraction interference pattern produced by a double slit.

5.5.1 Effect of Various Factors on Double-Slit Diffraction Pattern

A number of factors such as slit-width, slit separation, wavelength etc. may affect the double-slit diffraction pattern.

(i) If the wavelength λ of the incident light is increased, the envelope of the fringe pattern becomes broader and the fringes move further apart. The converse effect occurs on decreasing the wavelength.

(ii) If the slit width is increased the envelope of the fringe-pattern so changes that the central peak becomes sharper. The spacing between the fringes does not change as it depends on the slit separation. So within the central maximum, number of interference maxima is less. Decrease in slit-width causes the converse effect.

(iii) If the slit separation is increased, keeping the slit-width constant, the fringe become closer together, but the envelope of the pattern remains unaltered. So, more number of interference maxima will now fall within the central diffraction maximum.

5.5.2 Missing Order in Double Slit Diffraction Pattern

For single slit diffraction, the condition for minima of diffraction pattern is $a \sin \theta = \pm m\lambda$.

On the other hand, the double slit interference pattern has maxima at angle satisfying the condition

$$(a + b) \sin \theta = \pm n\lambda$$

An interesting situation arises when both equations are satisfied for a particular value of θ. This happens when,

$$\frac{a+b}{a} = \frac{n}{m} \quad [n \text{ and } m \text{ are both integers}]$$

In this case, nth order interference maximum will have zero intensity and will therefore be missing. In Fig. 5.5(c), $\frac{a+b}{a} = \frac{0.15}{0.05} = 3$, or $n = 3$ m. Hence 3, 6, 9 etc. order of interference maxima are absent which corresponds to 1, 2, 3 etc., order of diffraction dark bands. These are known as 'missing order' in the literature.

Thus, if the condition for maxima of interference pattern and minima of diffraction pattern are simultaneously satisfied for a given value of θ then the corresponding interference maxima will be missing or absent. *These orders are called missing orders in double-slit pattern as shown by the X marks in Fig. 5.5(c).*

5.6 DIFFERENCE BETWEEN SINGLE-SLIT AND A DOUBLE-SLIT DIFFRACTION PATTERN

1. The single-slit diffraction pattern consists of a central principal maximum with secondary maxima and minima on either side. The intensity of secondary maxima gradually decreases. The double-slit pattern, on the other hand, consists of a central maximum which contains equally spaced interference maxima and minima. The intensity of central diffraction maximum is four times the intensity of the central maximum due to diffraction at single slit.
2. The spacing of diffraction maxima and minima depends on the slit width a, but the spacing of the interference maxima and minima depends on both a and b, where b is the slit separation. The intensity

of the interference maxima and minima is not constant, but it decreases to zero on either side of the central maximum.

5.7 DIFFRACTION DUE TO PLANE DIFFRACTION GRATING

(a) Definition Diffraction grating is an arrangement consisting of a large number of parallel slits of the same width and separated by equal opaque space. It is thus a structure of alternate transparent and opaque space. Grating may be of two types:

1. transmission grating
2. reflection grating

A plane transmission diffraction grating is a well polished thin uniform optically plane transparent glass plate on which equidistant, exteremely close lines (rulings) are drawn with a sharp diamond point.

When the rullings are made on a polished metal surface, it is called as reflection grating. Here, we shall deal with a transmission grating.

On a grating the number of lines may vary usually from 2500 to 8000 per cm. Evidently to draw thousands of lines uniformly on a small space, the ruling machine must be very accurate and sensitive. Thus, a ruled grating is very costly. For ordinary work in a laboratory, replica gratings (commercial gratings) are used which are actually copies of an accurately ruled grating made on celluloid film by casting.

Grating provides a very valuable means of studying spectra. Each ruled line behaves as an opaque space while the transparent portion between two consequative ruled lines behaves as a slit. If a be the width of a clear space and b be the width of a ruled line, then the distance $(a + b = d)$ between the corresponding points is called **grating element** or **grating constant.**

(b) Theory The diffraction pattern is based on the superposition of Huygens' secondary wavelets which are supposed to be generated at every point on the wave front when it occupies the plane of the grating.

Let a parallel beam of monochromatic light be incident on a grating surface at an angle i. Figure 5.6(a) represents the section of the plane diffraction grating. Here S_1 S_2 is the plane of the grating perpendicular to the plane of the paper, a is the width of each slit and b is the width of each opaque space. The rays are

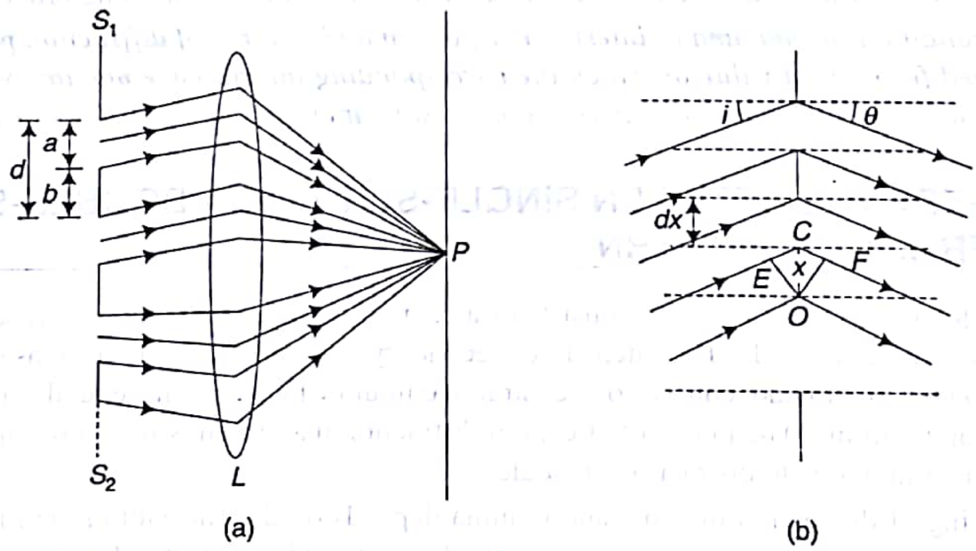


Fig. 5.6 (a) The section of the plane diffraction grating (b) Incident and diffracted rays through grating.

diffracted in all possible directions. Let us consider a beam of rays for which the angles of diffraction θ and L is a converging lens which focuses the diffracted rays on its focal plane.

As the source and the point of observation are effectively at infinity, obviously it is a case of Fraunhofer class of diffraction.

(c) Analytical/Mathematical Study of Intensity To study the diffraction phenomenon, let us consider a diffracting element of width dx at C a distance x from O, the midpoint of 1st slit (clear space).

Through grating let us draw incident and diffracted rays as shown in Fig. 5.6 (b). From C draw two perpendiculars OE and OF on the incident and corresponding diffracted rays. Then OE and OF represent incident and diffracted wave fronts. The equation of vibration on the incident wave front OE be represented by $y = Ae^{i\frac{2\pi}{\lambda}ct}$ where A is amplitude, λ is wavelength of the wave and c is the velocity of light.

So the equation of vibration on the diffracted wave front OF is given by

$$y = Ae^{i\frac{2\pi}{\lambda}(ct - ECF)} \qquad \ldots(5.14)$$

But path difference $ECF = EC + CF = x \sin i + x \sin \theta = x\varphi$ (say)

So the Eq. (5.14) reduces to $y = Ae^{i\frac{2\pi}{\lambda}(ct - x\varphi)}$

Now the disturbance at P due to element dx at C is given by

$$ds = y\,dx = Ae^{i\frac{2\pi}{\lambda}(ct - x\varphi)}\,dx$$

Hence the total disturbance at P due to entire clear space is given by

$$S_1 = \int ds = \int_{-a/2}^{a/2} A\,e^{i\frac{2\pi}{\lambda}(ct - x\varphi)}\,dx = Ae^{i\frac{2\pi}{\lambda}ct}\int_{-a/2}^{a/2} e^{-i\frac{2\pi}{\lambda}x\varphi}\,dx$$

Since the effect of amplitude is time independent quantity, the effective amplitude due to first clear space or slit is represented by

$$S_1' = A\int_{-a/2}^{a/2} e^{-i\frac{2\pi}{\lambda}x\varphi}\,dx$$

The disturbance due to second slit is given by

$$S_2 = Ae^{i\frac{2\pi}{\lambda}ct}\int_{d-a/2}^{d+a/2} e^{-i\frac{2\pi}{\lambda}x\varphi}\,dx$$

So, the effective amplitude due to second slit is given by,

$$S_2' = A\int_{d-a/2}^{d+a/2} e^{-i\frac{2\pi}{\lambda}x\varphi}\,dx = Ae^{-i\frac{2\pi}{\lambda}d\varphi}\int_{-a/2}^{a/2} e^{-i\frac{2\pi}{\lambda}x\varphi}\,dx$$

Similarly, the effective amplitude due to third slit is given by,

$$S_3' = Ae^{-i\frac{2\pi}{\lambda}2d\varphi}\int_{-a/2}^{a/2} e^{-i\frac{2\pi}{\lambda}x\varphi}\,dx$$

Proceeding in a similar way we can show that the effective amplitude due to Nth slit is given by

$$S_N' = Ae^{-i\frac{2\pi}{\lambda}(N-1)\,d\varphi}\int_{-a/2}^{a/2} e^{-i\frac{2\pi}{\lambda}x\varphi}\,dx$$

Hence the total effective amplitude due to all slits is given by

$$S' = S_1' + S_2' + S_3' + \ldots + S_N'$$

$$\therefore \quad S' = \left[A\int_{-a/2}^{a/2} e^{-i\frac{2\pi}{\lambda}x\varphi}dx\right]\left[1+e^{-i\frac{2\pi}{\lambda}d\varphi}+e^{-i\frac{2\pi}{\lambda}2d\varphi}+\ldots+e^{-i\frac{2\pi}{\lambda}(N-1)d\varphi}\right]$$

$$= \left[\frac{Aa\sin X}{X}\right]\left[\frac{1-e^{-i\frac{2\pi}{\lambda}Nd\varphi}}{1-e^{-i\frac{2\pi}{\lambda}d\varphi}}\right] \quad \text{where } X = \frac{\pi a\varphi}{\lambda}$$

$$= \left[\frac{Aa\sin X}{X}\right]\left[\frac{1-\left\{\cos\frac{2\pi}{\lambda}Nd\varphi - i\sin\frac{2\pi}{\lambda}Nd\varphi\right\}}{1-\left\{\cos\frac{2\pi}{\lambda}d\varphi - i\sin\frac{2\pi}{\lambda}d\varphi\right\}}\right]$$

Now the resultant intensity at P is proportional to the square of the amplitude. Let the constant of proportionality to be unity. The intensity at P is given by

$$I = \left[\frac{A^2a^2\sin^2 X}{X^2}\right]\left[\frac{\left(1-\cos\frac{2\pi}{\lambda}Nd\varphi + i\sin\frac{2\pi}{\lambda}Nd\varphi\right)^2}{\left(1-\cos\frac{2\pi}{\lambda}d\varphi + i\sin\frac{2\pi}{\lambda}d\varphi\right)^2}\right]$$

$$= \frac{A^2a^2\sin^2 X}{X^2}\left[\frac{2-2\cos^2\frac{2\pi}{\lambda}Nd\varphi}{2-2\cos^2\frac{2\pi}{\lambda}d\varphi}\right]$$

$$= \frac{A^2a^2\sin^2 X}{X^2}\left[\frac{\sin^2\frac{\pi Nd\varphi}{\lambda}}{\sin^2\frac{\pi d\varphi}{\lambda}}\right]$$

If we put $Y = \frac{\pi d\varphi}{\lambda}$, we can write

$$I = I_1\frac{\sin^2 NY}{\sin^2 Y} \qquad \ldots(5.15)$$

where $I_1 = \frac{A^2a^2}{X^2}\sin^2 X$ is the expression obtained due to single slit.

(d) Discussion From the expression (5.15) it is seen that there are two terms $\frac{\sin^2 X}{X^2}$ and $\frac{\sin^2 NY}{\sin^2 Y}$ which control the intensity of the diffracted pattern. The first expression gives the diffraction pattern produced by single slit and the second term gives the effect of interference due to N identical slits.

(e) Conditions for Maxima and Minima In order to find maxima and minima of the intensity of interference fringes we must have,

$$\frac{dI}{dY} = 0$$

Now,

$$\frac{dI}{dY} = I_1\left[\frac{\sin^2 Y\cdot 2(\sin NY)N\cos NY - \sin^2 NY(2\sin Y\cos Y)}{\sin^4 Y}\right]$$

or,

$$\frac{dI}{dY} = 2I_1\left[\frac{N\sin NY\cos NY}{\sin^2 Y} - \frac{\sin^2 NY\cos Y}{\sin^3 Y}\right]$$

or, $$\frac{dI}{dY} = 2I_1 \left(\frac{\sin^2 NY}{\sin^2 Y}\right)(N \cot NY - \cot Y) \qquad \text{...(5.16)}$$

Hence, for maxima or minima,

either, $$\frac{\sin^2 NY}{\sin^2 Y} = 0 \quad \text{or} \quad N \cot NY - \cot Y = 0 \Rightarrow N \cot NY = \cot Y$$

when $\sin NY = 0$ $\sin Y \neq 0$ and $N \cot NY \neq \cot Y$, one gets $\frac{d^2I}{dY^2} > 0$ (i.e., positive).

So, $\sin NY = 0$ gives the condition of minimum

when $N \cot NY = \cot Y = 0$ and $\sin Y = 0$, one gets $\frac{d^2I}{dY^2} < 0$ (i.e., negative).

So, $N \cot NY = \cot Y$ and $\sin Y = 0$ gives the condition of maximum

(f) Principal Maxima When N is very large one obtains intensity maxima at

$$\sin Y = 0 \text{ i.e., } Y = \pm m\pi \quad [m = 0, 1, 2, \ldots\text{etc.}]$$

i.e., $$\frac{\pi d \varphi}{\lambda} = \pm m\pi \quad \text{or,} \quad \frac{\pi}{\lambda}(a + b) \sin \theta = \pm m\pi \quad \text{[For normal incidence]}$$

Hence $\quad (a + b) \sin \theta = \pm m\lambda$

These are known as the principal maxima.

For $Y = m\pi$, $\frac{\sin NY}{\sin Y} = \frac{0}{0}$ = indeterminate. But in limit

$Y \to m\pi$ we get

$$\lim_{Y \to m\pi} \frac{\sin NY}{\sin Y} = \lim_{Y \to m\pi} \frac{N \cos NY}{\cos Y} \quad \text{[using } L. \text{ Hospital's rule]}$$

Hence the corresponding resultant intensity is given by

$$I_p = N^2 I_1 \qquad \text{...(5.17)}$$

Thus the intensity of principal maxima increases as the number of slits (N) increases, but due to the presence of the factor $\frac{\sin^2 X}{X^2}$, whose value decreases with the increase of the angle of diffraction (θ), the intensity of principal maxima decreases with the increase in the order number of bands.

(g) Secondary Minima From the second condition we get zero intensity i.e., $I = 0$ when $\sin NY = 0$ provided $\sin Y \neq 0$. This gives, $NY = \pm p\pi$ [where p is an integer except O, N, $2N$, $3N$ …etc., in these cases $\sin Y = 0$ and we get principal maxima]. So $\sin NY = 0$ gives $NY = \pm p\pi$.

or, $$\frac{N\pi(a + b) \sin \theta}{\lambda} = \pm p\pi \quad \text{[for normal incidence } i = 0]$$

or, $$(a + b) \sin \theta = \pm \frac{p\lambda}{N}$$

such that $p \neq sN$ [where $s = 0, 1, 2, 3, \ldots$etc.]

Therefore, $$(a + b) \sin \theta = \pm \frac{\lambda}{N}, \pm \frac{2\lambda}{N}, \ldots \pm \frac{(N-1)\lambda}{N}, \pm \frac{(N+1)\lambda}{N}$$

Omitting the values $(a + b) \sin \theta = 0, \pm \frac{N\lambda}{N}, \pm \frac{2N\lambda}{N}, \ldots$

which are principal maxima.

It is obvious that between two principal maxima there are $(N-1)$ equally spaced minima. Hence, there should be $(N-2)$ secondary maxima in between these minima.

(h) Secondary Maxima These maxima are weak in intensity. Intensity of these maxima are given by

$$I_S = I_1 \frac{\sin^2 NY}{\sin^2 Y} = \frac{I_1}{\sin^2 Y \operatorname{cosec}^2 NY}$$

$$= \frac{I_1}{\sin^2 Y\,[1+\cot^2 NY]}$$

$$= \frac{1}{\sin^2 Y\left[1+\dfrac{\cot^2 Y}{N^2}\right]} \qquad \text{[From the condition } N\cot NY = \cot Y\text{]}$$

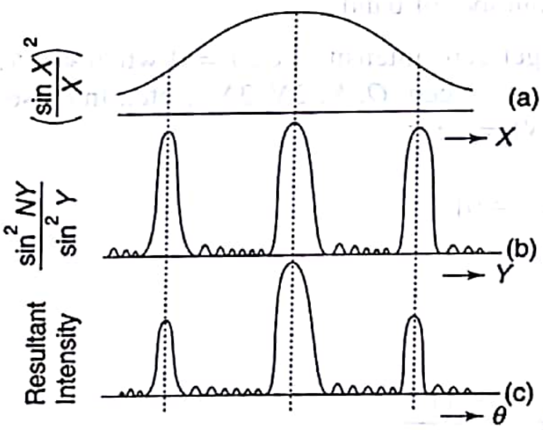
or,
$$I_S = \frac{N^2 I_1}{\sin^2 Y\left[N^2+\dfrac{\cos^2 Y}{\sin^2 Y}\right]} = \frac{N^2 I_1}{N^2\sin^2 Y+\cos^2 Y}$$

$$= \frac{N^2 I_1}{(N^2-1)\sin^2 Y+1} = \frac{I_P}{(N^2-1)\sin^2 Y+1}$$

Therefore
$$\frac{I_S}{I_P} = \frac{1}{1+(N^2-1)\sin^2 Y} \qquad \text{...(5.18)}$$

Thus greater the N-value, weaker are the secondary maxima. In practice, N is very large in a grating and secondary maxima are too weak to be generally visible in grating.

In Fig (5.7) curves are drawn by plotting $\left(\dfrac{\sin X}{X}\right)^2$ against X [Fig. 5.7(a)], $\dfrac{\sin^2 NY}{\sin^2 Y}$ against Y [Fig. 5.7 (b)] and also by plotting the product of them against θ [Fig. 5.7(c)]. The intensities of secondary maxima fall of between two adjacent principal maxima and are unequally spaced. They also lack symmetry, the lack being greatest adjacent to principal maxima towards which their peaks get shifted. The intensity distribution for $N = 5$ is shown in Fig. 5.7(d).



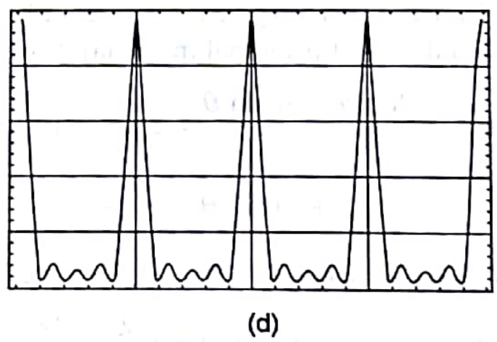


Fig. 5.7 (a) Diffraction pattern due to single slit (b) Interference effect due to secondary waves from the N slits (c) the resultant intensity parttern (d) Intensity distribution due to 5 slits

If white light is used, instead of monochromatic light, each wavelength will produce its own maxima. So each maxima will consist of a band of seven colours.

Now, the angle of diffraction $\theta \propto \lambda$, wavelength of light. So, angle of diffraction for red will be greater than for violet. It means that inner edge of each maxima (i.e., the edge facing the central maximum) will be violet and the outer edge red.

5.7.1 Condition for Absent Spectra

For the mth order of principal maximum in the direction θ, we have the condition

$$(a + b) \sin \theta = m\lambda \quad \text{[for normal incidence } i = 0] \qquad \ldots(5.19)$$

If a be such that the pth order diffraction mininium occurs in the same direction θ, then

$$a \sin \theta = p\lambda \qquad \ldots(5.20)$$

If these two conditions are satisfied simultaneously then mth order principal maximum will be absent from the resulting spectra

From Eq. (5.19) and (5.20), we get

$$\frac{a+b}{a} = \frac{m}{p} \qquad \ldots(5.21)$$

If $b = a$ then from Eq. (5.21), $m = 2p = 2, 4, 6 \ldots (\because p = 1, 2, 3\ldots)$ which implies that the second, fourth, sixth ... etc. order of principal maxima will be absent corresponding to the diffraction minima $p = 1, 2, 3, \ldots$

5.7.2 Ghost Lines

In case of a perfect (or ideal) grating the rulings must be equi-spaced, but practically no grating is perfect. This imperfectness in rulings of a diffraction grating causes some errors. When these errors are random, the grating gives a continuous background illumination. On the other hand, if they are progressive in nature, then one may get sharper spectral lines in planes other than the local plane of the grating. But most frequently the errors repeat themselves periodically because of some defect in the driving mechanism of the machines which make rulings in the diffraction gratings. This results in the generation of false lines along with the principal maxima of a perfect (ideal) g.ating. These false lines are called ghost lines.

5.7.3 Overlapping of Spectral Lines

If the incident light has a large range of wavelengths, then the lines of shorter wavelength and higher order may overlap on lines of longer wavelength and lower order.

For a grating, having grating element $(a + b)$, the angle of diffraction θ in the mth order spectrum for wavelength λ is given by

$$(a + b) \sin \theta = m\lambda$$

Now for a given grating $a + b$ = constant. So θ will be the same i.e., overlapping will occur if $m\lambda$ = constant. For instance, red line of 700 nm in the 3rd order will overlap with green line of 525 nm in 4th order, violet line 420 nm in 5th order etc., since $3 \times 700 = 4 \times 525 = 5 \times 420 = \ldots =$ constant.

5.7.4 Maximum Number of Orders Formed by Diffraction Grating

The condition for principal maxima is given by

$$(a + b) \sin \theta = m\lambda$$

or, $$m = \frac{(a+b)\sin\theta}{\lambda}$$

The number of order will be maximum when $\sin\theta = 1$

$\therefore$ $$(m)_{\max} = \frac{a+b}{\lambda}$$

If $(a+b) < 2\lambda$; then $(m)_{\max} < 2$

This means that for normal incidence, only first order of spectrum will be present, when $(a+b) < 2\lambda$.

5.7.5 Condition of Minimum Deviation of Rays

For oblique incidence of a parallel beam of light of wavelength λ, the grating equation is

$$(a+b)(\sin i + \sin\theta) = m\lambda \qquad \text{...(5.22)}$$

Rewriting Eq. (5.22)

$$2(a+b)\sin\frac{i+\theta}{2}\cos\frac{i-\theta}{2} = m\lambda$$

Now, the deviation suffered by the rays is $\delta = i + \theta$, so that

$$2(a+b)\sin\frac{\delta}{2}\cos\frac{i-\theta}{2} = m\lambda$$

So $\sin\frac{\delta}{2}$ and hence δ, will have minimum value when $\cos\frac{i-\theta}{2}$ is maximum

or, $$i - \theta = 0$$

or, $$i = \theta$$

which is the required condition.

Thus, the rays forming spectrum in a grating suffer minimum deviation when the angle of incidence equal the angle of diffraction.

5.7.6 Angular Dispersive Power of a Grating

The angular dispersive power of a grating is defined as the rate of change of the angle of diffraction θ with the change in wavelength λ of light. It is thus $\frac{d\theta}{d\lambda}$.

The diffraction of the mth order principal maximum for a wavelength λ and angle of diffraction θ is given by

$$(a+b)\sin\theta = m\lambda \quad \text{[For normal incidence } i = 0\text{]}$$

Differentiating it with respect to λ we write

$$(a+b)\cos\theta\, d\theta = m d\lambda$$

$\therefore$ $$\frac{d\theta}{d\lambda} = \frac{m}{(a+b)\cos\theta} = \frac{ms}{\cos\theta} \qquad \text{...(5.23)}$$

where $s = \frac{1}{a+b}$ = number of lines per unit length of grating

Equation (5.23) is the expression for the angular dispersive power of a grating

Here $d\theta$ is the angular separation between two lines with difference in wavelengths as $d\lambda$.

The Eq. (5.23) gives the following conclusions:

(i) The dispersive power is directly proportional to the order of spectrum (m)

(ii) The dispersive power is inversely proportional to the grating element ($a + b$)

(iii) The dispersive power is inversely proportional to cos θ. i.e., the larger the value of θ, higher is the dispersive power.

5.8 PRISM AND GRATING SPECTRUM

Prism Spectrum	*Grating Spectrum*
(i) It is formed by dispersion.	(i) It is formed by diffraction.
(ii) The prism spectrum is bright.	(ii) The grating spectra are fainter as the order increases.
(iii) The prism forms one spectrum.	(iii) The grating forms spectra of many orders.
(iv) The prism spectrum depends on the material of the prism.	(iv) The grating spectra does not depend on the material of the grating.
(v) In most cases prism spectrum is not pure.	(v) The grating spectra are pure.
(vi) The resolving power of a prism is small.	(vi) The resolving power of the grating is large.

5.9 RAYLEIGH'S CRITERION OF RESOLUTION

Lord Rayleigh proposed a criterion for the resolution of two near by objects. According to him "The two nearby point objects are said to be just resolved by the optical system, if the position of the central maximum of diffraction pattern of one coincides with the first minimum of the diffraction pattern of the other".

It is also applicable to the resolution of spectral lines of equal intensity formed by prism or grating spectrograph. In case of spectral lines, the criterion says, the angular separation between the principal maxima of two spectral lines in a given order, should be equal to half angular width of either maximum.

To illustrate the above criterion, consider the diffraction pattern of the two spectral lines of wavelengths λ and $\lambda + d\lambda$ and let A and B are the central maximum due to them respectively [Fig. 5.8(a)]. When the difference in the angle of diffraction is large, the two images are seen as separate.

In Fig. 5.8(b) the central maxima corresponding to the wavelengths λ and $\lambda + d\lambda$ are very close and therefore the two images overlap and they cannot be distinguished as separate images.

In Fig. 5.8(c), the position of central maxima of A corresponding to wavelength λ, coincide with the position of first minima corresponding to wavelength $\lambda + d\lambda$. Similarly, the central maxima of B coincides with the first minima of A. The resultant intensity curve shows a dip at C in the middle of the central maxima of A and B. It is estimated that the intensity at C is about 20 percent less than that at A or B. The position of central maxima of A and B coincides with the first minima of the diffraction pattern of each other and it satisfies the Rayleigh criterion. The spectral lines are, therefore said to be just resolved.

The smallest separation (linear or angular) between the two point objects at which they appear just separated is called the limit of resolution of an optical instrument and the reciprocal of the limit of resolution is called **resolving power**.

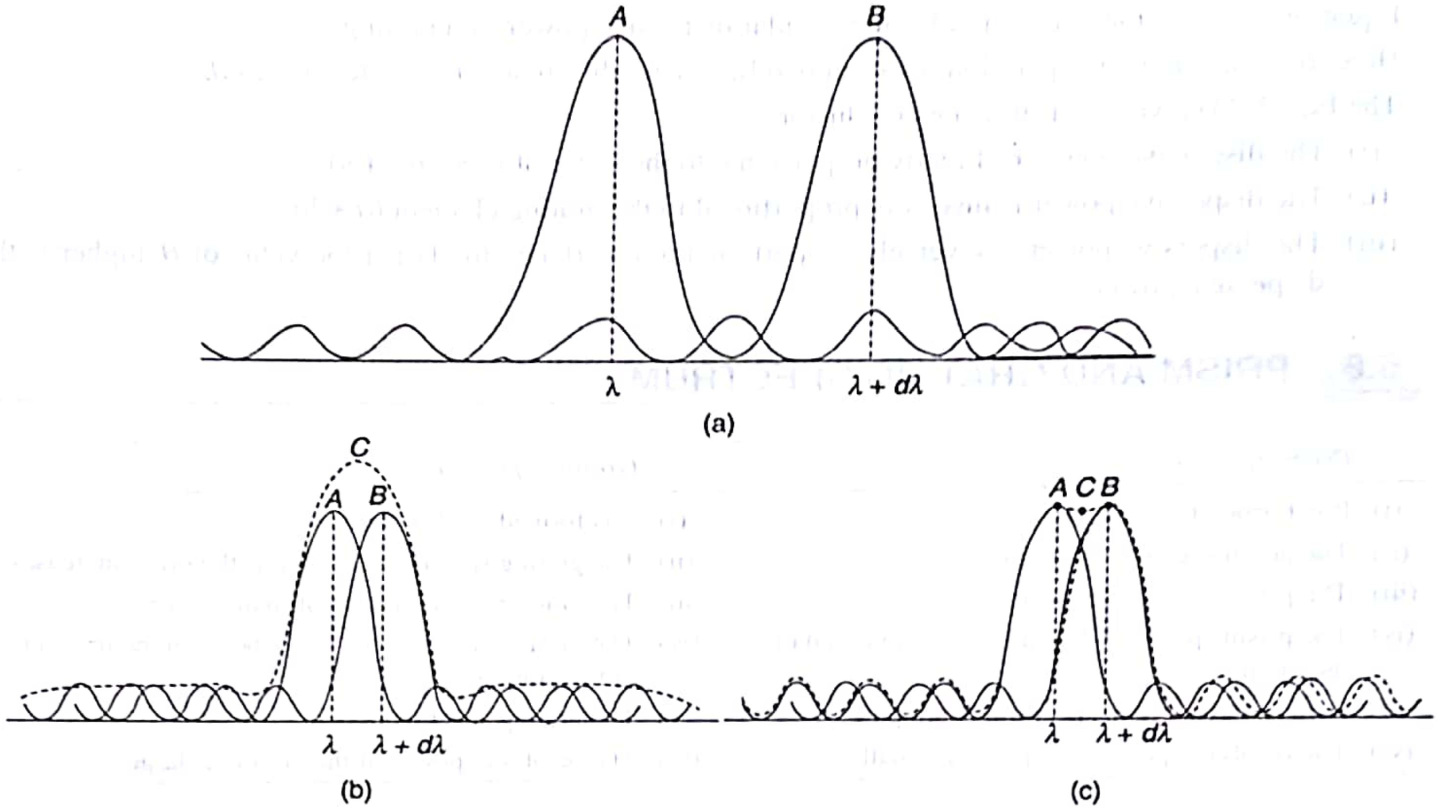


Fig. 5.8 Rayleigh Criterion of Resolution.

5.10 RESOLVING POWER OF A GRATING

The resolving power (RP) of a plane grating is its ability to just distinguish (resolve) two nearby spectral lines with two close wavelengths.

If the wavelength of two nearby spectral lines be λ and $\lambda + d\lambda$, the resolving power of the grating is defined mathematically as $\frac{\lambda}{d\lambda}$, $d\lambda$ being the smallest wavelength difference for which the spectral lines can be just resolved at wavelength λ.

The mth order principal maximum of a spectral line of wavelength λ of a grating for normal incidence is

$$(a + b)\sin\theta = m\lambda \qquad \ldots(5.24)$$

where $(a + b)$ is the grating element, θ the diffraction angle corresponding to mth order.

The same for wavelength $\lambda + d\lambda$ will be

$$(a + b)\sin(\theta + d\theta) = m(\lambda + d\lambda) \qquad \ldots(5.25)$$

According to Rayleigh criterion, these two spectral lines will be just resolved, when the principal maximum of one falls exactly over the first points of minimum intensity of the other line. This is possible if the extra path difference introduced is $\frac{\lambda}{N}$, where N is the total number of lines of the grating surface.

$$(a + b)\sin(\theta + d\theta) = m\lambda + \frac{\lambda}{N} \qquad \ldots(5.26)$$

Equating the right-hand sides of the Eqs. (5.25) and (5.26)

$$m(\lambda + d\lambda) = m\lambda + \frac{\lambda}{N}$$

$$\therefore \quad \frac{\lambda}{d\lambda} = mN \qquad ...(5.27)$$

which is required expression for the resolving power of a grating.

From Eq. (5.24), we have

$$m = \frac{(a+b)\sin\theta}{\lambda}$$

$$\therefore \quad \frac{\lambda}{d\lambda} = mN = \frac{N(a+b)\sin\theta}{\lambda} \qquad ...(5.28)$$

$$= W\frac{\sin\theta}{\lambda} \qquad ...(5.29)$$

where $W = N(a + b)$, the total width of the ruled surface of the grating.

For maximum resolving power $\sin\theta = 1$

Hence maximum, $RP = \dfrac{W}{\lambda}$

The relation $\dfrac{\lambda}{d\lambda} = Nm$ shows that RP increases with

(i) the order of the spectrum

(ii) increases with total number of lines in the effective part of the grating

(iii) resolving power does not depend on grating element

5.11 RESOLVING POWER OF MICROSCOPE

The primary function of a microscope is not to magnify an object but to reveal those finer details in the object which are invisible to unaided eye. The limit of resolution of a microscope is measured by the least permissible linear distance between two point objects so that the two images appear just resolved. **The reciprocal of the limit of resolution is called the resolving power of the microscope.**

Let us consider two point objects are separated by a distance x and they are very close to the objective and normal to the axis of the microscope. If α is the semi-vertical angle of the cone of the rays by the objective of the microscope then it can be shown that the two image will appear just resolved if,

$$2x\sin\alpha = 1.22\,\lambda \qquad ...(5.30)$$

or, $$x = \frac{1.22\,\lambda}{2\sin\alpha}$$

which is the required limit of resolution. The high resolving power microscope are generally oil immersion type in which the space between the objective of the microscope and the object is filled with an oil of refractive index μ.

Then $$x = \frac{1.22\,\lambda}{2\mu\sin\alpha} \qquad ...(5.31)$$

where $\mu\sin\alpha$ is known as numerical aperture (N_A) of the objective.

Resolving power, $$RP = \frac{1}{x} = \frac{2\mu\sin\alpha}{1.22\,\lambda} \qquad ...(5.32)$$

By using ultraviolet light with smaller λ and quartz lenses the resolving power of the microscope can also be increased. Such microscope is called ultra microscope. At present, higher resolving power is obtained in electron microscope where electrons are used instead of white light.

Worked-out Examples

Example 5.1 A light of wavelength 6000 Å falls normally on a straight slit of 0.10 mm width. Calculate the total angular width of the central maxima.

Sol. The condition for 1st order diffraction minima is

$$a \sin \theta = \lambda \quad \text{[Here } \theta \text{ is the half angular width of the central maxima]}$$

Here $a = 0.10$ mm $= 0.01$ cm, $\lambda = 6.0 \times 10^{-5}$ cm

$$\therefore \qquad \sin \theta = \frac{\lambda}{a} = \frac{6.0 \times 10^{-5}}{0.01} = 6.0 \times 10^{-3}$$

Since θ is very small, then $\sin \theta \simeq \theta = 6.0 \times 10^{-3}$ radians

The angular width of the central maxima is

$$= 2\theta$$

$$= 2 \times 6 \times 10^{-3} = 1.2 \times 10^{-2} \text{ radians.}$$

Example 5.2 Light of wavelength 5000 Å is incident normally on a slit. The first minima of the diffraction pattern is observed to lie at a distance of 5 mm from the central maxima on a screen placed at a distance of 2 m from the slit. Calculate the width of the slit.

Sol. For 1st order minima $a \sin \theta = \lambda$

If θ be small, then $\sin \theta \simeq \theta$. Hence

$$a\theta = \lambda \quad \text{or,} \quad \theta = \frac{\lambda}{a} = \frac{5000 \times 10^{-8}}{a} \qquad \ldots(1)$$

Further, θ is also given by $\theta = \dfrac{x}{D}$

where x is the position of first maximum from the central maximum and D is the distance of screen from the slit.

Here $x = 0.5$ cm and $D = 200$ cm.

So $$\theta = \frac{0.5}{200} \text{ radian} \qquad \ldots(2)$$

From Eqs. (1) and (2), we have $\dfrac{5000 \times 10^{-8}}{a} = \dfrac{0.5}{200}$

or, $$a = \frac{5000 \times 10^{-8} \times 200}{0.5} = \frac{5 \times 2 \times 10^{-2}}{5}$$

$$= 0.02 \text{ cm}$$

So the width of the slit is 0.02 cm.

Example 5.3 A convex lens of 40 cm focal length is employed to focus the Fraunhofer diffraction pattern of a single slit of 0.3 mm width. Calculate the linear distance of the first-order dark band from the central band. The wavelength of the light is 589 nm. [WBUT 2008]

Sol. For nth dark band

$$a \sin \theta = n\lambda$$

Here $a = 0.3$ mm $= 0.3 \times 10^{-3}$ m; $n = 1$; $\lambda = 589 \times 10^{-9}$ m

and $\sin \theta \simeq \theta$ for very small θ

So $$\theta = \frac{\lambda}{a} = \frac{589 \times 10^{-9}}{0.3 \times 10^{-3}} \text{ rad.} = 196.3 \times 10^{-5} \text{ radians}$$

$\therefore$ linear distance $x = \theta \times f = 196.3 \times 10^{-5} \times 40 \times 10^{-2}$ m

$= 0.7852$ mm.

Example 5.4 A single slit is illuminated by light composed of two wavelengths, λ_1 and λ_2. One observes that due to Fraunhofer diffraction, the first minimum obtained for λ_1 coincides with the second diffraction minimum of λ_2. What is the relation between λ_1 and λ_2?

Sol. For wavelength λ_1; the position of first minimum ($n = 1$) is

$$a \sin \theta_1 = \lambda_1 \quad \ldots(1)$$

For wavelength λ_2; the position of second minimum ($n = 2$) is

$$a \sin \theta_2 = 2\lambda_2 \quad \ldots(2)$$

It is given that the direction of first minimum (θ_1) due to λ_1 coincides with the second minimum due to λ_2

i.e. $$\theta_1 = \theta_2 = \theta$$

So from Eqs. (1) and (2)

$$a \sin \theta = \lambda_1 = 2\lambda_2$$

$$\therefore \quad \lambda_1 = 2\lambda_2$$

Example 5.5 Deduce the missing orders for a double-slit Fraunhofer diffraction pattern if the slit width are 0.16 mm and they are 0.8 mm apart. [WBUT 2008]

Sol. Missing orders are obtained when interference maximum and diffraction minimum corresponds to the same value of direction angle θ, i.e.,

$$(a + b) \sin \theta_n = n\lambda \quad \text{[interference maximum]}$$

and $$a \sin \theta_m = m\lambda \quad \text{[diffraction minimum]}$$

So, $$\frac{a+b}{a} = \frac{n}{m}$$

Here $a = 0.16$ mm and $b = 0.8$ mm

$$\therefore \quad \frac{n}{m} = \frac{0.16 + 0.8}{0.16} = \frac{0.96}{0.16} = 6$$

For values of $m = 1, 2, 3$ etc. or $n = 6, 12, 18$, etc.

So, the 6th, 12th, 18th etc. orders of interference maxima will be missing in the diffraction pattern.

Example 5.6 In a double-slit Fraunhofer diffraction, calculate the fringe spacing on the screen 50 cm away from the slit if they are illuminated with blue light of wavelength 4800 Å. Slit separation $b = 0.1$ mm and slit width $a = 0.020$ mm.

Sol. The fringe spacing can be obtained by the formula $d = \dfrac{\lambda D}{(a+b)}$

Here $a = 0.02$ mm $= 0.002$ cm, $b = 0.1$ mm $= 0.01$ cm

$$\lambda = 4800 \text{ Å} = 4800 \times 10^{-8} \text{ cm}, \quad D = 50 \text{ cm}$$

$$\therefore \quad d = \frac{4800 \times 10^{-8} \times 50}{0.012} \text{ cm}$$

$$= 0.2 \text{ cm}$$

So the fringe spacing is 0.2 cm.

Example 5.7 A parallel beam of light of wavelength (5890 Å) falls normally on a plane transmission grating having 4250 lines/cm. Find the angle of diffraction for maximum intensity in first order. [WBUT 2006]

Sol. We known that $(a + b) \sin \theta = n\lambda$

or, $$\sin \theta = \frac{n\lambda}{(a+b)}$$

Here $n = 1$ and $\dfrac{1}{(a+b)} = 4250$ lines/cm

$$\lambda = 5890 \text{ Å} = 5890 \times 10^{-8} \text{ cm}$$

Therefore, $$\sin \theta = 5890 \times 10^{-8} \times 4250$$

$$= 0.25$$

$$\therefore \quad \theta = 14.5°$$

Example 5.8 Light of wavelength 5000 Å is incident normally on a plane transmission grating. Find the difference in deviations in the first and third order spectra. The number of lines per cm on the grating surface is 6000.

Sol. Given $\lambda = 5000$ Å $= 5 \times 10^{-5}$ cm, $\dfrac{1}{a+b} = 6000$

For the first-order spectrum

$$(a + b) \sin \theta_1 = \lambda \quad [\text{Here } n = 1]$$

or, $$\sin \theta_1 = \frac{\lambda}{a+b} = 5 \times 10^{-5} \times 6{,}000 = 0.30$$

or, $$\theta_1 = 17.5°$$

For the third-order spectrum

$$(a + b) \sin \theta_3 = 3\lambda \quad [\text{Here } n = 3]$$

$$\sin \theta_3 = \frac{3\lambda}{a+b} = 3 \times 5 \times 10^{-5} \times 6000$$

$$= 0.9$$

or, $\theta_3 = 64.2°$

Now the difference in deviation is $\theta_3 - \theta_1 = 64.2° - 17.5° = 46.7°$

Example 5.9 Light of wavelength $\lambda = 500$ nm falls on grating normally. Two adjacent principal maxima occur at $\sin\theta = 0.2$ and $\sin\theta = 0.3$ respectively. Calculate the grating element.

Sol. The position of the principal maximum of order n is given by

$$(a + b)\sin\theta = n\lambda$$

The two adjacent orders, say n and $n + 1$ occur at $\sin\theta = 0.2$ and $\sin\theta = 0.3$ respectively.

Thus $(a + b) \times 0.2 = n\lambda$ and $(a + b) \times 0.3 = (n + 1)\lambda$

Subtracting, we get

$$(a + b) \times 0.1 = \lambda = 500 \text{ nm} = 500 \times 10^{-9} \text{ m}$$
$$= 5000 \times 10^{-8} \text{ m}$$

So, the grating element is

$$(a + b) = 5 \times 10^{-4} \text{ cm.}$$

Example 5.10 How many orders will be visible if the wavelength of the incident light is 5000 Å and the number of lines per inch on the grating is 2620?

Sol. We know that $(a + b)\sin\theta = n\lambda$

For maximum order to be possible, $\theta = 90°$

$$\sin\theta = \sin 90° = 1$$

Given that $(a + b) = \dfrac{2.54}{2620}$ cm; $\lambda = 5000 \text{ Å} = 5000 \times 10^{-8}$ cm

Hence $$n = \frac{a + b}{\lambda} = \frac{2.54}{2620 \times 5 \times 10^{-5}}$$

or, $$n > 19$$

So, the highest order of spectrum possible =19.

Example 5.11 In a grating spectrum, which spectral line in 5th order will overlap with 4th order line of 5890 Å?

Sol. The grating equation for principal maximum is,

$$(a + b)\sin\theta = n\lambda \quad \text{...(1)}$$

For 5th order spectrum, $(a + b)\sin\theta = 5\lambda$ ...(2)

Here λ is wavelength of the 5th spectral line

The wavelength of 4th order spectrum is 5890 Å, hence

$$(a + b)\sin\theta = 4 \times 5890 \times 10^{-8} \quad \text{...(3)}$$

The two spectral lines overlap, so θ is same.

Equating Eqs. (2) and (3), we get

$$5\lambda = 4 \times 5890 \times 10^{-8}$$

or, $$\lambda = \frac{4 \times 5890 \times 10^{-8}}{5} = 4712 \times 10^{-8} = 4712 \text{ Å}$$

So the wavelength of the 5th order spectral line is 4712 Å.

Example 5.12 How many orders will be observed by a grating having 4000 lines per cm if it is illuminated normally by light of wavelength in the range 5000 Å to 7500 Å.

Sol. The maximum order visible with grating is

$$n_{\max} = \frac{a+b}{\lambda}$$

Given, $(a+b) = \dfrac{1}{4000}$ Hence $n_{\max} = \dfrac{1}{4000\,\lambda}$

For $\lambda = 5000\text{ Å} = 5 \times 10^{-5}$ cm $\quad n_{\max} = \dfrac{1}{4000 \times 5 \times 10^{-5}} = 5$

and for $\lambda = 7500\text{ Å}$ $\quad n_{\max} = \dfrac{1}{4000 \times 7.5 \times 10^{-5}} = 3.3$

$= 7.5 \times 10^{-5}$ cm

So, the observed orders range between 3 to 5.

Example 5.13 What is the minimum number of lines of a grating which resolve the 3rd order spectrum of two lines having wavelengths of 5890 Å and 5896 Å. [WBUT 2004]

Sol. Given: $\lambda_1 = 5890$ Å; $\lambda_2 = 5896$ Å; $d\lambda = 6$ Å

The average wavelength $\lambda = \dfrac{\lambda_1 + \lambda_2}{2} = 5893$ Å and $n = 3$

The resolving power $(\mathrm{R_P}) = \dfrac{\lambda}{d\lambda} = \dfrac{5893 \times 10^{-8}}{6 \times 10^{-8}} = 982$

Further $\mathrm{R_P} = nN$ [N is the number lines on the grating]

or, $nN = 982$

or, $3N = 982$ or, $N = \dfrac{982}{3} = 327.33 \simeq 327$

So the minimum number of lines in the grating will be 327.

Example 5.14 Calculate the least width that a grating must have to resolve two components of the sodium D-line in the second order, the grating having 800 lines/cm. The wavelength for D_1 and D_2 lines of sodium are 5893 Å and 5896 Å respectively. [WBUT 2008]

Sol. We know that

the number of slits on the grating = width of the total number of lines × number of lines per cm.

$= x \times 800$ [Here x is the width]

Given $\lambda_1 = 5893$ Å and $\lambda_2 = 5896$ Å

So $d\lambda = 3$Å and average $\lambda = \dfrac{\lambda_1 + \lambda_2}{2} = 5894.5$ Å

The $\mathrm{R_P} = \dfrac{\lambda}{d\lambda} = \dfrac{5894.5}{3} = 1964.83 \simeq 1965$

Further $\mathrm{R_P} = nN = 1965$

or, $2 \times N = 1965$

or, $N = \dfrac{1965}{2} = 982.5$

To obtain least width $x \times 800 = 982.5$

$$x = \frac{982.5}{800} \text{ cm} = 1.228 \text{ cm}$$

Example 5.15 A microscope is used to resolve two equally bright points separated by 5.55×10^{-5} cm. What is the numerical apperture (NA) of the objective if light of wavelength 5461 Å is used?

Sol. The limit of resolution of the microscope is

$$x = \frac{1.22\,\lambda}{2\,\mu \sin \alpha}$$

Here $\mu \sin \alpha$ is called numerical apperture (NA),

$$\lambda = 5461\ 10^{-8} \text{ cm}$$

$$x = 5.55 \times 10^{-5} \text{ cm}$$

So, $$\mu \sin \alpha = \frac{1.22 \times 5461 \times 10^{-8}}{2 \times 5.55 \times 10^{-5}} = 0.60$$

So, Numerical apperture of the objective is 0.60.

Example 5.16 The microscope objective gathers light over a cone of semi-angle 30° and uses visible light of average wavelength 5500 Å. Estimate the resolving limit of microscope.

Sol. The resolving limit of microscope is given by

$$x = \frac{1.22\,\lambda}{2 \sin \theta}$$

Here $\lambda = 5500$ Å $= 5500 \times 10^{-8}$ cm, $\theta = 30°$

So, $$x = \frac{1.22 \times 5500 \times 10^{-8}}{2 \sin 30°} = \frac{1.22 \times 5500 \times 10^{-8}}{2 \times 0.5} = 6.71 \times 10^{-5} \text{ cm}$$

Example 5.17 A diffraction grating, which has 4000 lines/cm is used at normal incidence. Calculate the dispersive power of the grating in the third order spectrum in the wavelength region 5000A°

Sol. Here $\lambda = 5000 \times 10^{-8}$ cm; $n = 3$

$$s = 4000 \text{ lines/cm}$$

$\therefore$ $$a + b = \frac{1}{4000} = 2.5 \times 10^{-4} \text{ cm}$$

Now $$\sin \theta = \frac{n\lambda}{a+b} = \frac{3 \times 5000 \times 10^{-8}}{2.5 \times 10^{-4}} = 0.6$$

$$\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - (0.6)^2} = 0.8$$

$\therefore$ Dispersive power $$\frac{d\theta}{d\lambda} = \frac{ns}{\cos \theta} = \frac{3 \times 4000}{0.8} = 15000$$

Review Exercise

Part 1: Multiple Choice Questions

1. In diffraction, the size of the obstacle should be such that it
   (a) is comparable to the wavelength of the light used
   (b) is greater than the wavelength of the light used
   (c) has no connection with the wavelength of the light used
   (d) none of these
2. In Fraunhofer diffraction, the incident wave front is
   (a) plane (b) cylindrical (c) spherical (d) none of these
3. Double-slit interference pattern is the limiting case of double-slit diffraction pattern when the
   (a) distance between the slits tends to zero
   (b) distance of the source and the slits tends to infinity
   (c) slit widths tend to zero
   (d) distance between the slits tends to infinity. {WBUT 2008]
4. Fraunhofer diffraction arises when the source of light and screen is effectively at
   (a) finite distance (b) infinite (c) semi-finite (d) none of these
5. In a Fresnel diffraction the source of light is effectively at
   (a) finite (b) infinite
   (c) both finite and infinite (d) none of these
6. In Fraunhofer diffraction minima are
   (a) all perfectly dark (b) never perfectly dark
   (c) perfectly bright (d) none of these
7. If the wavelength of the light used in single-slit diffraction is increased then the width of the central maximum
   (a) decreases (b) increases (c) remains same (d) none of these
8. The intensity of central maximum due to double slit diffraction pattern is ______ times greater than that of single slit pattern.
   (a) eight (b) three (c) four (d) two
9. The intensity of principal maximum in the spectrum of a grating with N number of lines is proportional to
   (a) $\frac{1}{N}$ (b) N (c) N^2 (d) $\frac{1}{N^2}$ [WBUT 2008]
10. Missing order in the interference maximum in Fraunhofer double-slit pattern occurs if the
    (a) slit width is decreased
    (b) slit width is constant, but the slit separation is increased
    (c) slit width is increased
    (d) none of these.

11. Angle of diffraction of the first order maximum in a diffraction grating
    (a) increases if the separation between the ruling is increased
    (b) decreases if the separation between the ruling is increased
    (c) does not change if the separation between the ruling is increased
    (d) change depends on the wavelength of the incident light [WBUT 2006]
12. The resolving power of a grating, having N number of total rulings, in nth order is
    (a) $\frac{n}{N}$ (b) nN (c) $\frac{N}{n}$ (d) none of these
13. With the increase of number of lines per centimetre of plane transmission grating
    (a) both dispersive power and resolving power decreases
    (b) both dispersive power and resolving power increases
    (c) dispersive power decreases but resolving power increases
    (d) dispersive power increases but resolving power decreases.
14. A diffraction pattern is obtained using a beam of red light. What happens if the red light is replaced by blue light?
    (a) Bands disappear
    (b) Bands becomes broader and farther apart
    (c) No change
    (d) Diffraction bands became narrower and crowded together.

Answers

1. (a)	2. (a)	3. (c)	4. (a)	5. (a)	6. (b)	7. (b)	8. (c)
9. (c)	10. (b)	11. (b)	12. (b)	13. (b)	14. (d)		

Short Questions with Answers

1. Distinguish between Fresnel and Fraunhofer classes of diffraction.

In Fresnel diffraction phenomena either the source or the point of observation or both are at finite distances from the diffraction obstacle. Here, the incident wave front is either spherical or cylindrical.

In Fraunhofer diffraction phenomena both the source and the point of observation are effectively at infinite distance from the diffraction obstacle. Here the incident wave front is plane.

2. What is the effect of (a) increasing the slit width (b) increasing the slit separation and (c) increasing the wavelength of light in a double slit diffraction pattern?

(a) Effect of increasing the slit width: If we increase the slit width the central peak becomes sharp. The spacing between the fringes does not change as it depends on the slit separation so within the central maximum, number of interference maximum is less. Reverse effect happens if we decrease the slit width.

(b) Effect of increasing the slit separation: If the slit separation is increased, keeping the slit width constant, the fringe become closer together, but the envelope of the pattern remains unaltered. So, more number of interference maxima will be observed within the central diffraction maximum.

(c) *Effect of increasing wavelength of light:* If the wavelength λ of the incident light is increased, the envelope of the fringe pattern becomes broader and fringe move further apart.

3. **What is missing order?**

   Refer to Article 5.5.2.

4. **What is the difference between single-slit and double-slit diffraction pattern?**

   Refer to Article 5.6.

5. **What is replica grating?**

   Good transmission grating could be made by taking a cast of the ruled surface with some transparent material and such casts are called replica gratings. Replica gratings give satisfactory performance where the highest resolving power is not needed.

6. **What is the Rayleigh criterion?**

   According to the Rayleigh criterion the maximum resolving power of an optical instrument corresponds to the condition that the principal maximum of a diffraction pattern of one point object falls exactly on the first minimum of the diffraction pattern of an adjacent point object.

7. **What is the limit of resolution of an optical instrument?**

   The smallest separation (linear or angular) between the two point objects at which they appear just separated is called the limit of resolution of an optical instrument and the reciprocal of the limit of resolution is called its resolving power.

8. **What is the difference between grating spectra and prism spectra?**

   Refer to Article 5.8.

9. **In a plane transmission grating 15000 lines/inch are taken. Why?**

   With the increase in the number of lines the secondary maxima relative to the principal maxima decreases and becomes negligible. When N becomes large enough the secondary maxima are not visible with a grating having about 15000 lines or more per inch.

Part 2: Descriptive Questions

1. What is meant by diffraction of light? Distinguish between Fresnel and Fraunhofer classes of diffraction [WBUT 2005]
2. (i) Explain the difference between interference and diffraction

   (ii) Derive the expression of intensity at a point for Fraunhofer diffraction due to double slit. Draw the intensity distribution carve (diffraction pattern) and explain it [WBUT 2006]
3. The intensity distribution for single-slit diffraction is

$$I = I_0\left(\frac{\sin^2\beta}{\beta^2}\right) \quad \text{where } \beta = \frac{\pi b \sin\theta}{\lambda}$$

   b is the width of the slit and λ is the wavelength of light. Show that secondary maxima are given by the equations

$$\tan\beta = \beta$$

   [WBUT 2006]
4. What do you mean by (i) ghost lines, (ii) absent spectra.
5. What is the condition for the missing order spectra for a diffraction grating? What particular order will be absent if width of the slit is equal to width of opaque space of a grating?

6. Plot a intensity distribution curve for Fraunhofer diffraction due to single slit and find the positions of maxima and minima.
7. What is grating element? What is its relation with the number of rulings?
8. What is meant by the resolving power of an optical instrument? Explain Rayleigh's criterion for just resolution.
9. What do you understand by resolving power of a grating? Derive the necessary expression.
10. Define the resolving power of microscope. Deduce an expression for it.
11. Obtain an expression for the resultant intensity in a single-slit Fraunhofer diffraction process and show the intensity pattern graphically. [WBUT 2005]
12. Show that the intensity of secondary maxima formed by single-slit Fraunhofer diffraction process is nearly 4.5% of the principal maximum. [WBUT 2004]
13. State Rayleigh criterion of resolution and discuss its significance in studying spectral lines.

Part 3: Numerical Problems

1. A single slit form diffraction pattern of Fraunhofer class with white light. The second maximum for red light of wavelength 6500 Å coincides with the third maximum of an unknown wavelength. Find the unknown wavelength. **[4642 Å]**
2. What is the highest order spectrum which may be observed with monochromatic light of wavelength 5000 Å by means of a grating using 5000 lines cm^{-1}? [Ans. **4]**
3. Find the missing orders for a double-slit Fraunhofer pattern if the width of each slit is 0.15 mm and they are separated by a distance of 0.60 mm. **[5, 10, 15 etc.]**
4. A diffraction grating 2 cm wide is just able to resolve sodium D-lines in second order. Find the number of rulings per mm. **[24.5]**
5. A plane transmission grating has 15,000 lines per inch. Find
   (a) the resolving power of grating, and
   (b) the smallest wavelength difference that can be resolved with a light of wavelength 6000 Å in the second order. **[(a) 3×10^4 (b) 20 Å]**
6. Calculate the angles at which the first dark band and the next bright band are formed in the Fraunhofer diffraction pattern of a slit 0.3 mm wide ($\lambda = 5890$ Å). **[$\theta = 6'$; $\theta = 12'$]**
7. Calculate the number of lines per cm of a grating which gives an angle of diffraction equal to 30° in the first order of light of wavelength 600 nm. **[8333.3]**
8. A microscope is used to resolve two point sources separated by a distance 4×10^{-5} cm. If the wavelength of light be 5461 Å, calculate the numerical aperture of the objective. **[1.665 cm]**

CHAPTER

6

Polarization of Light

6.1 INTRODUCTION

A wave is a disturbance which can propagate through a medium. While propagating through the medium, the wave cause the particles of the medium to vibrate either along the direction of propagation of the wave or in a direction perpendicular to that of propagation. **The wave which causes the particles of the medium to vibrate along the direction of propagation of it is known as longitudinal wave and the wave which causes vibration of the particles of medium in a perpendicular direction is called transverse wave.** Light is a form of electromagnetic wave.

The phenomena of interference and diffraction of light were successful beyond doubt to prove that light is a form of wave. But they failed to decide wheather light is a transverse or longitudinal wave. The fact that light is a transverse wave was established only after the discovery of a phenomenon of light which is known as polarization. It was Huygens who first discovered the phenomenon of polarization in 1690. The phenomena of interference and diffraction can be exhibited by all types of waves but polarization can be exhibited by the transverse waves only. So, it was the discovery of polarization of light that helped us to prove that light is a transverse wave.

6.2 MECHANICAL DEMONSTRATION OF POLARIZATION

Analogy is very much important in physics. It helps one to understand the physical phenomena easily through comparison. To understand polarization clearly, let us take the help of an analogous mechanical illustration.

Let us consider a horizontal string *AB* attached to a fixed point *B* at one end [Fig. 6.1]. Holding its other end *A* at hand, one can generate transverse waves in many different planes along the string. The particles of the string will vibrate perpendicularly to the direction of propagation of the waves. Let us repeat the experiment in the presence of two parallel slits S_1 and S_2 between points *A* and *B* as shown in the diagram. Amongst all the generated waves, only one wave, which is contained by the plane passing through the pair of parallel slits S_1 and S_2, will be able to pass through up to the point *B*.

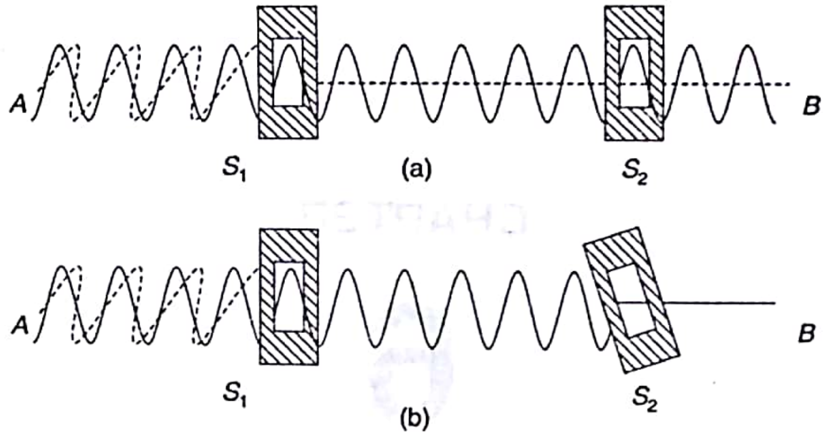


Fig. 6.1 Polarization of mechanical wave such a wave containing vibrations confined in a given plane, is called a polarized wave

The polarized wave passes through the parallel slit S_2. But if the slit S_2 is turned and made perpendicular to S_1 as shown in the Fig. 6.1(b), no wave is obtained beyond S_2. If the string AB is replaced by a spring of very small cross-section and longitudinal waves are generated along it, then turning of the slit S_2 through 90° [Fig. 6.1(b)] makes no difference to the wave. In the changed situation also, it can pass through both the slits S_1 and S_2 undisturbed. This shows that a longitudinal wave cannot be polarized.

6.3 SYMBOLIC REPRESENTATION OF UNPOLARIZED AND POLARIZED LIGHT

In Section 6.2 we have observed that the wavelength of the wave to be polarized should be comparable to the length of the slit. The wavelength of light waves is very small, so it is required to have slits of comparable length for getting polarization of light. It is difficult to make such small slits artificially. But some crystals provide us such slits for getting polarized beam of light. Before discussing polarization by crystals, let us see how unpolarized and polarized beams of light are represented in symbolic form. Light is considered as a transverse electromagnetic wave which consists of two mutually perpendicular vectors. One of them is the electric vector and the other is the magnetic vector. The direction of propagation of the light wave is also perpendicular to each of the aforesaid vectors. In an unpolarized or natural light beam the electric and magnetic vectors keep on vibrating continuously. The average effect of these two random vibrations gives rise to the impression that in unpolarized light, vibrations of the electric and magnetic vectors are symmetrically distributed in their plane which is perpendicular to the direction of propagation of light (Fig. 6.2).

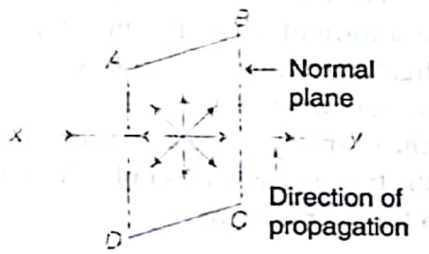


Fig. 6.2 Vibration is normal to the direction of propagation

A light beam in which the electric and magnetic vectors vibrate symmetrically about the direction of propagation of it, is called unpolarized light. On the other hand, a light beam in which the electric and magnetic vectors do not vibrate symmetrically about the direction of propagation of it, is called polarized light. The conventions of drawing unpolarized and polarized light rays are given below: (i) If the vibrations of the polarized light beam take place in the plane of the paper, then they are represented by small straight lines perpendicular to the direction of propagation with two opposite arrow-heads [Fig. 6.3(a)], (ii) If

the vibrations of the polarized beam take place in a plane perpendicular to the plane of the paper, then they are represented by small dots all along the direction of propagation of light [Fig. 6.3.(b)]. (iii) An unpolarized beam of light has vibrations in all possible directions in a plane perpendicular to the direction of propagation. Each vibration in this case may be considered as the resultant of two mutually perpendicular component vibrations. And for this reason, an unpolarized beam is usually represented by the simultaneous use of dots and small straight lines along with two opposite arrow-heads [Fig. 6.3(c)].

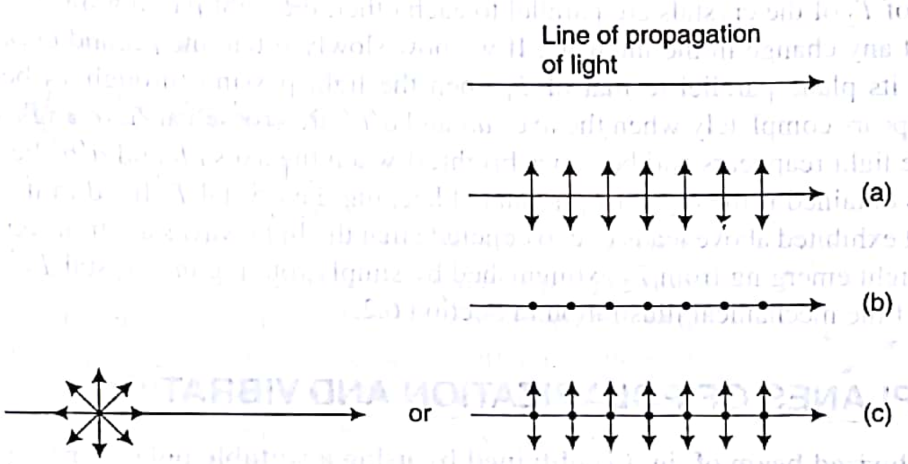


Fig. 6.3 Symbolic representation of polarized and unpolarized light

6.4 EXPERIMENT OF POLARIZATION WITH TOURMALINE CRYSTAL

Let us now experimentally exhibit polarization of light with the help of tourmaline crystal. It is made up of silicates of various metals and boron. It is a natural crystal with some interesting optical effects. It is a hexagonal crystal and almost transparent with a greenish tinge. The greatest diagonal of the hexagonal is known as its optic axis (the diagonal *ab* in the crystal of Fig. 6.4).

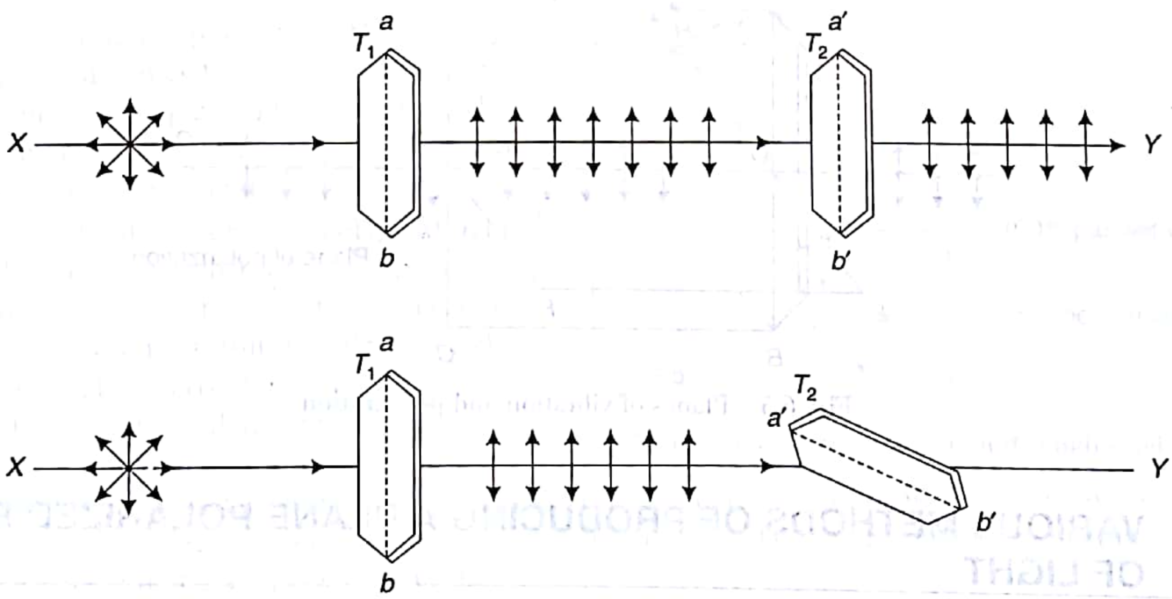


Fig. 6.4 Polarization of light by tourmaline crystal

As can be seen in the Fig. 6.4, T_1 is a thin piece of tourmaline crystal. A narrow beam of light moving along the direction *XY* is normally incident on a flat surface of the crystal. A part of the light transmits through the crystal. No change in the intensity of the emergent light is observed except slight tinge in it. The colour depends upon the nature of the crystal. There is no change in the intensity of the transmitted beam of light if the crystal T_1 is rotated about the line *XY* as axis. Let us now allow the transmitted light to fall on another similar crystal T_2 placed behind the first one and observe the light passing through T_1 and T_2. When the axes *ab* of T_1 and $a'b'$ of T_2 of the crystals are parallel to each other, the light passing through T_2 appears slightly darker but without any change in the intensity. If we now slowly rotate the second crystal T_2 about the line of vision keeping its plane parallel to that of T_1, then the light passing through T_2 becomes dimmer and ultimately it disappears completely when the axes *ab* and $a'b'$ are crossed at right angles. If the crystal T_2 is rotated further, the light reappears and becomes brightest when the axes *ab* and $a'b'$ become parallel again. The same result is obtained if the crystal T_1 is rotated keeping the crystal T_2 fixed in its initial position. The simple experiment exhibited above leads one to conclude that the light waves are transverse waves, otherwise we could not get light emerging from T_2 extinguished by simply rotating the crystal T_2. A similar effect was observed in case of the mechanical illustration in Section 6.2.

6.5 THE PLANES OF POLARIZATION AND VIBRATION

When a plane polarized beam of light is obtained by using a suitable polarizer (e.g., a calcite crystal), the vibrations of the electric vectors are confined in a plane. Such a plane is called plane of vibration. The plane perpendicular to the plane of vibration is called the plane of polarization. The plane of vibration contains the vibrating electric vectors. In Fig. 6.5, a polarized beam of light transmitted through a tourmaline crystal *AB* has been shown. The plane *ABCD* contains both, the vibrations of the electric vectors of the polarized beam and the direction of propagation *OQ*. The plane *EFGH* represents the plane of polarization. The line *OQ*, which represents the direction of propagation, also lies on this plane.

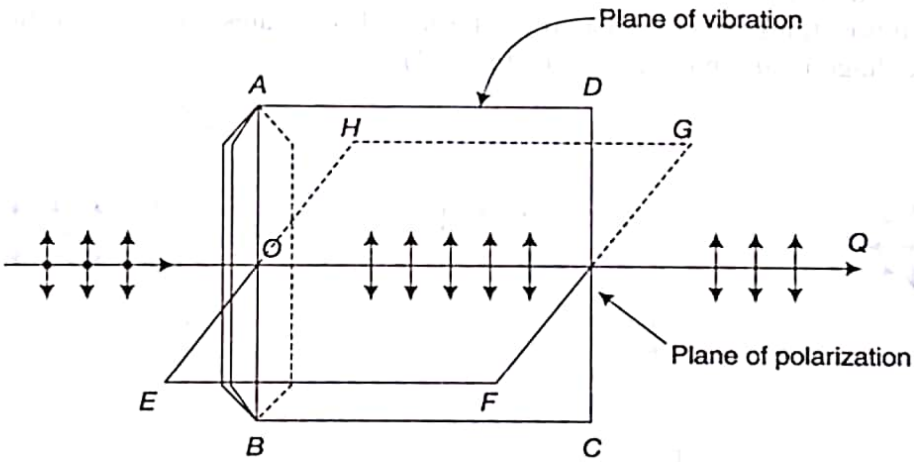


Fig. 6.5 Planes of vibration and polarization

6.6 VARIOUS METHODS OF PRODUCING A PLANE POLARIZED BEAM OF LIGHT

A beam of light in which the vibrations of electric vectors are confined in a plane is called a plane polarized beam. In the previous section, we have seen how a plane polarized beam of light can be obtained

by using a tourmaline crystal. There are many other methods to obtain a plane polarized beam of light. Let us discuss some of them in brief.

(a) Polarization by reflection
(b) Polarization by refraction
(c) Polarization by double refraction
(d) Polarization by selective absorption
(e) Polarization by scattering

6.6.1 Polarization by Reflection

The simplest method of obtaining a polarized beam of light is that of reflection. In 1808 Malus discovered that a polarized beam of light can be obtained if ordinary light is reflected by a transparent reflecting surface like a plane sheet of glass. He also found that the degree of polarization depends on the angle of incidence of light. He further observed that at a particular angle (which depends on the nature of the reflecting surface) the reflected beam becomes totally polarized. **This angle of incidence at which totally polarized beam of light is obtained is called the angle of polarization of the reflecting surface.** The angle of polarization for glass is about 57°. Figure 6.6 shows complete polarization by reflection at the polarizing angle θ_p.

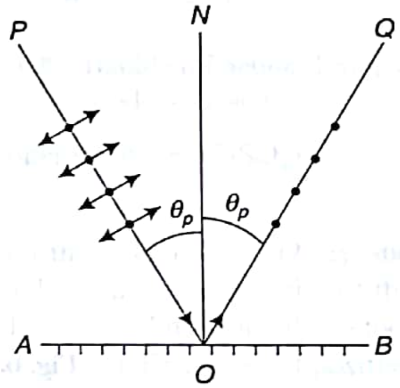


Fig. 6.6 Polarization by reflection

6.6.2 Polarization by Refraction

When a beam of unpolarized light is refracted through a transparent medium, the refracted light becomes partially polarized. To obtain a completely polarized beam of light, one has to get the beam refracted through a number of transparent refracting plates, which are placed in such a way that they become parallel to each other having been separated by an air gap. The more the number of plates, the more will be the purity of the polarized light.

For example, if a beam of unpolarized light be incident on a number of glass plates at polarizing angle, then the reflected ray is completely plane polarized. But the refracted light is not completely polarized, rather it is partially polarized. The refracted ray retains cent per cent parallel vibration and a reduced percentage of normal vibration. The normal vibration can further be reduced if the beam is repeatedly passed through more parallel plates. Hence degree of polarization can be increased by increasing the member of plates which are parallel to the first glass plate (Fig. 6.7).

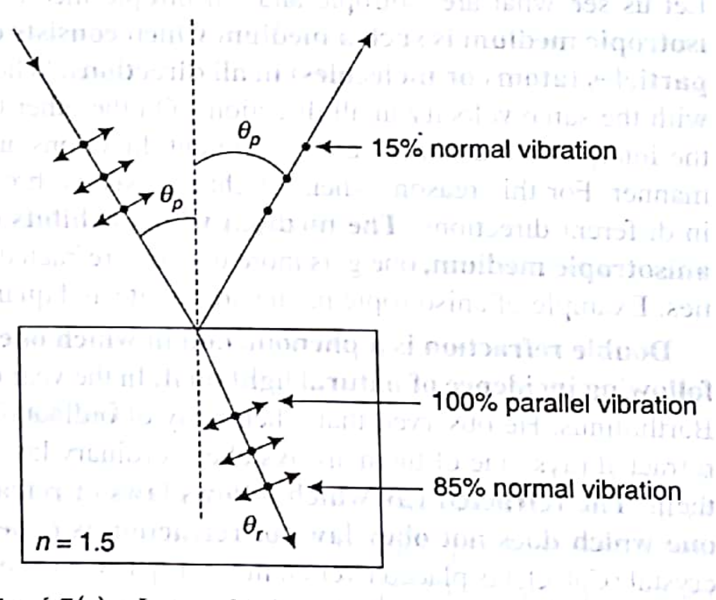


Fig. 6.7(a) Incomplete polarization through single refraction.

Scanned by CamScanner

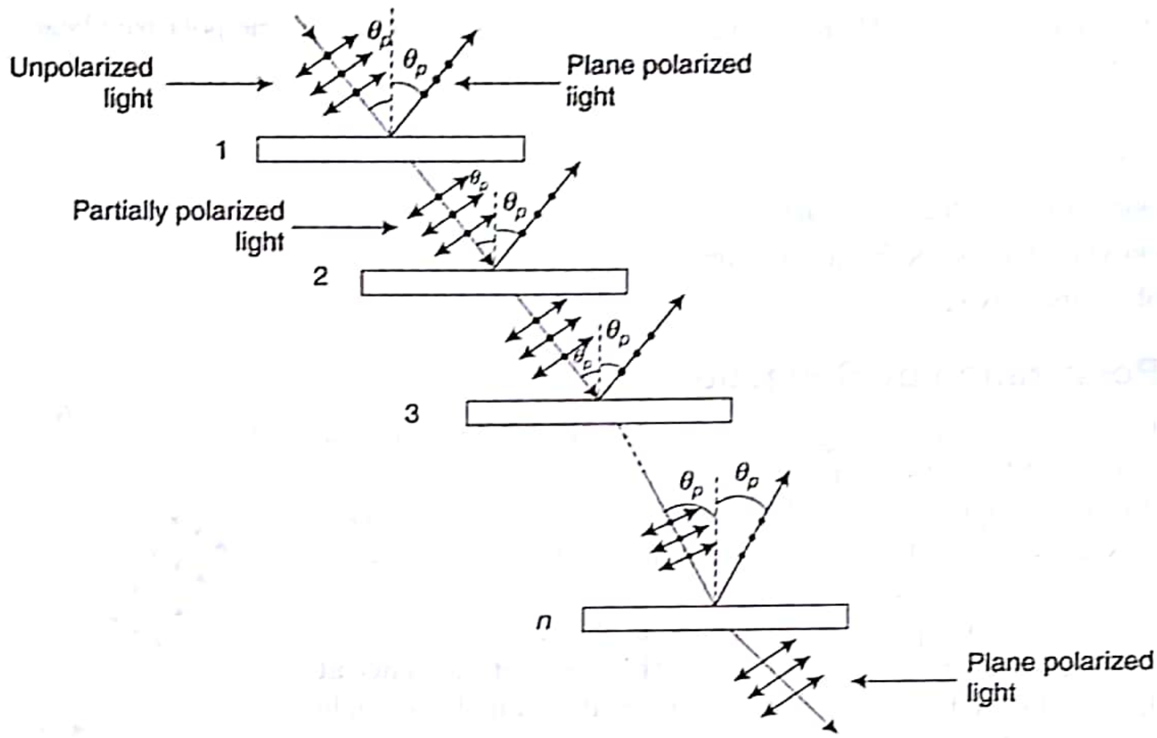


Fig. 6.7(b) Complete polarization through multiple refraction

6.6.3 Polarization by Double Refraction or Bifringence

Let us see what are isotropic and anisotropic media before starting the analysis of double refraction. **An isotropic medium is such a medium which consists of a periodic, regular and identical arrangement of particles (atoms or molecules) in all directions.** When a light ray passes through such a medium, it travels with the same velocity in all directions. On the other hand, there are many a crystalline substance in which the interparticle distances along different directions are different. The particles are arranged in an irregular manner. For this reason, when a light ray passes through such a medium, the velocity of light is different in different directions. **The medium which exhibits different velocities in different directions is called anisotropic medium,** one gets more than one refracted rays because of the presence of more than one velocities. Example of anisotropic media are calcite and quartz crystals.

Double refraction is a phenomenon in which one gets two refracted rays from a refracting medium following incidence of natural light on it. In the year of 1669, double refraction was discovered by Erasmus Bartholinus. He observed that when a ray of ordinary light is incident on a calcite crystal, it splits into two refracted rays, one of them always obeys ordinary laws of refraction and other one, in general may not obey them. **The refracted ray which follows laws of refraction is called ordinary ray (*O*-ray) and the other one which does not obey laws of refraction is called extra-ordinary ray (*E*-ray).** In Fig. 6.8, a calcite crystal ($CaCO_3$) is placed over an ink-dot (P) on a paper. Looking through the crystal one can see two images of the dot. If a glass slab is placed instead of the calcite crystal, one can see only one image. The two images seen through the calcite crystal evidently represent two different emergent rays. Hence, we get evidence of double refraction. **The crystals like calcite which give rise to double refraction are called bifringent crystals and the phenomenon is called bifringence.** The refractive index of a substance is related to the velocity of light in the concerned medium and is given by the following relation:

$$\mu = \frac{c}{v} \qquad ...(6.1)$$

The refractive index for the ordinary ray remains same in all directions but that for the extraordinary ray varies from one direction to another. Hence, the velocity of the extraordinary ray is variable $\left(\text{as } \mu = \frac{c}{v} \text{ and } c \text{ is the velocity of light in free space}\right)$. **For one direction, the refractive indices for the ordinary and the extra ordinary rays are equal. This direction is called the optic axis of the crystal.** There are two types of crystals; in one type there exists only one direction in which the velocities of the two refracted rays (O-ray and E-ray) remain same. This type of crystal is known as **uniaxial crystal.** In another type of crystal there exist two such directions in which the O-ray and the E-ray have same velocities. Such crystals are known as **biaxial crystals.** Uniaxial crystals have only one optic axis while biaxial crystals have two optic axes. Examples of uniaxial crystals are calcite ($CaCO_3$) and quartz (SiO_2), that of biaxial crystals are aragonite and coper sulphate ($CaSO_4$).

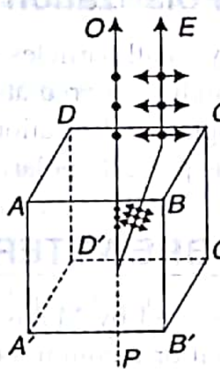


Fig. 6.8 Double refraction through a calcite crystal.

Positive and Negative Crystals

The anisotropic crystals can be categorized into two categories depending on the velocities of O-ray and E-ray inside the crystal: (i) positive crystal, and (ii) negative crystal. The crystal in which the velocity of the O-ray (v_o) is greater than that of the E-ray (v_e) is called the positive crystal and the crystal in which the velocity of E-ray (v_e) is greater than that of O-ray (v_o) is called the negative crystal (Fig. 6.9). Examples of positive crystals are quartz and iron oxide and that of negative crystals are calcite and tourmaline crystals.

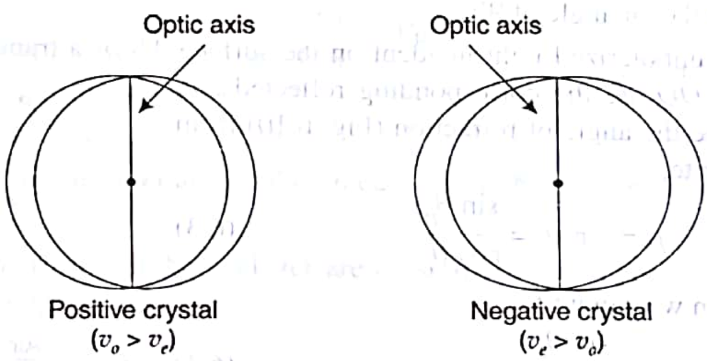


Fig. 6.9 Positive and negative crystals.

6.6.4 Polarization by Selective Absorption

If a substance absorbs one or more wavelengths from the several wavelengths present in a beam of light, then the absorption is called selective absorption. As for example, a transparent red coloured substance absorbs all colours except the red. Tourmaline is one of the several substances which produce plane polarized beam of light by selective absorption. We have observed in Section 6.4 that when unpolarized light is transmitted through a thin piece of tourmaline crystal, the transmitted beam of light becomes plane polarized with vibrations parallel to the optic axis of the crystal. The unpolarized light on entering the crystal splits into two polarized components—one known as ordinary ray whose vibrations are perpendicular to the optic axis of the crystal and the other—known as extra-ordinary ray with vibrations parallel to the optic axis. The crystal absorbs the O-ray and transmits the E-ray causing polarization through selective absorption.

6.6.5 Polarization by Scattering

When very small particles scatter a beam of light, the resulting scattered light is found to be partially polarized. For light scattered at an angle of 90°, the degree of polarization is the greatest. But this is usually far from complete polarization. Light from the blue sky, being the result of scattering by small atmospheric particles, is partially polarized.

6.7 BREWSTER'S LAW

It was observed by Malus that when a beam of unpolarized light is incident on the surface of a transparent material or medium, both the reflected and refracted beams are partially plane polarized. By changing the angle of incidence, Malus observed that for a particular value of the angle of incidence, the reflected light beam becomes completely plane polarized, the plane of vibration being at right angles to the plane of incidence. **This angle for which the reflected beam is completely polarized is known as angle of polarization.**

Brewster's Law Brewster obserbed that there exists a simple relation between the angle of polarization (θ_p) and the refractive index (μ) of the reflector relative to the surrounding medium. This relation is given by

$$\mu = \tan\theta_p \qquad ...(6.2)$$

The relation (6.2) is known as Brewster's Law. The polarizing angle for glass is 57°.

It can be proved from Brewster's law that when light is incident at the polarizing angle, the reflected beam and the refracted beam make an angle of 90°.

Let PO be a beam of unpolarized light incident on the surface AB of a transparent medium at angle of polarization θ_p, OR and OQ are the corresponding reflected and refracted beams. Let θ_r be the angle of refraction (Fig. 6.10) from Brewster's law we can write,

$$\mu = \tan\theta_p = \frac{\sin\theta_p}{\cos\theta_p} \qquad ...(6.3)$$

From Snell's law, again we can write

$$\mu = \frac{\sin\theta_p}{\sin\theta_r} \qquad ...(6.4)$$

Now, from Eqs. (6.3) and (6.4), we can have

$$\frac{\sin\theta_p}{\cos\theta_p} = \frac{\sin\theta_p}{\sin\theta_r}$$

which implies, $\cos\theta_p = \sin\theta_r = \cos(90° - \theta_r)$

$\therefore$ $\theta_p = 90° - \theta_r$

or, $\theta_p + \theta_r = 90° \qquad ...(6.5)$

Hence, $\theta = 90° \quad [\because \theta_p + \theta_r + \theta = 180°]$

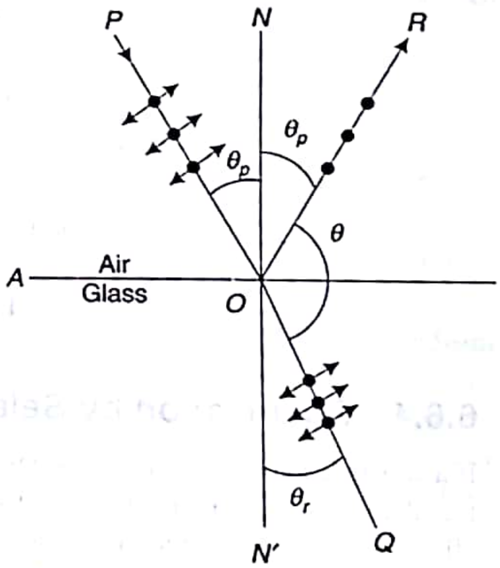


Fig. 6.10 Incidence at an angle of polarization results in $\theta_p + \theta_r = 90°$

6.8 MALUS' LAW

To discuss about this law one requires the concept of polarizer and analyzer. Let us define polarizer and analyzer.

Polarizer A polarizer is an optical device which can generate a polarized beam of light when an unpolarized beam of light is incident on it.

Analyzer An analyzer is an optical device which can determine whether an incident beam of light is polarized or not when the same is incident on it.

While working experimentally with polarization of light in 1809, E L Malus observed that the intensity of light transmitted by the analyzer varies with the angle between the plane of transmission of the analyzer and that of the polarizer. This is called Malus' law.

Statement of Malus' Law This law states that if a completely plane polarized light beam is incident on an analyzer, then the intensity of the light emerged varies directly as the square of the cosine of the angle between the plane of transmission of the polarizer and that of the analyzer. Let a be the amplitude of the light transmitted by the polarizer and θ be the angle of the plane of transmission of the polarizer with that of the analyzer (Fig. 6.11)

Let us now resolve 'a' into two components; $a \cos \theta$ and $a \sin \theta$, respectively along the plane of transmission of the analyzer and a direction perpendicular to it. One of these two components (i.e., $a \sin \theta$) is eliminated and the other component (i.e., $a \cos \theta$) is freely transmitted through the analyzer. Hence, the intensity of light emerging from the analyzer is given by

$$I_\theta = a^2 \cos^2 \theta$$

or,

$$I_\theta = I \cos^2 \theta \quad \text{... (6.6)}$$

where $I = a^2$ is the intensity of polarized light incident on the analyzer. The Eq. (6.6) is the Malus law.

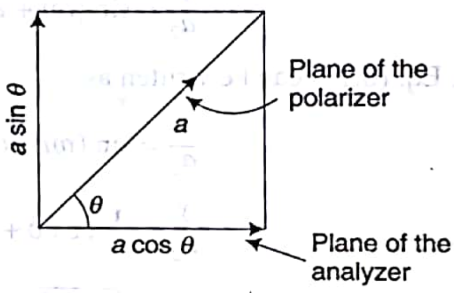


Fig. 6.11 Resolving of the amplitude a of the plane polarized light.

Two special cases of Malus' law are given below:

(i) If the polarizer and the analyzer are parallel to each other, then we get $\theta = 0°$ or $\theta = 180°$
So, $I_\theta = I$.

(ii) If, however, the polarizer and the analyzer are crossed (i.e., the angle between them is 90°), then $\theta = 90°$, this implies $I_\theta = 0$.

6.9 CLASSIFICATION OF POLARIZED LIGHT

So far we have discussed about plane polarized light only. Depending on the nature of the polarized light, it can be classified as (i) linearly polarized, (ii) circularly polarized, and (iii) elliptically polarized light. In case of circularly polarized light, the electric vector vibrates in a fixed plane along a straight line. In case of circularly polarized light, the electric vector rotates along a circle without changing its magnitude. And in case of elliptically polarized light, the electric vector rotates along an ellipse. The magnitude of the electric vector varies between a maximum and a minimum values which are respectively the semi-major and semi-minor axes of the ellipse. Circularly and elliptically polarized light waves can be generated by superposing two mutually perpendicular linearly polarized light waves. When the amplitudes of the two superposing waves are equal, the resultant light wave becomes circularly polarized and when they are unequal, the resultant light wave becomes elliptically polarized.

6.10 PRODUCTION OF PLANE, CIRCULARLY AND ELLIPTICALLY POLARIZED LIGHT BY ANALYTICAL METHOD

Let us consider two orthogonal light vectors (i.e., electric vectors) of frequency $\nu \left(= \frac{\omega}{2\pi}\right)$ and amplitudes a_1 and a_2 given by the following two equations:

$$x = a_1 \sin(\omega t) \quad ...(6.7)$$

$$y = a_2 \sin(\omega t + \delta) \quad ...(6.8)$$

Rearranging Eqs. (6.7) and (6.8), we get

$$\frac{x}{a_1} = \sin(\omega t) \quad ...(6.9)$$

and

$$\frac{y}{a_2} = \sin(\omega t + \delta) \quad ...(6.10)$$

Now, Eq. (6.10) can be written as

$$\frac{y}{a_2} = \sin(\omega t)\cos\delta + \cos(\omega t)\sin\delta$$

or,

$$\frac{y}{a_2} = \frac{x}{a_1}\cos\delta + \sqrt{1 - \frac{x^2}{a_1^2}}\sin\delta$$

or,

$$\left(\frac{y}{a_2} - \frac{x}{a_1}\cos\delta\right) = \sqrt{1 - \frac{x^2}{a_1^2}}\sin\delta$$

Now squaring both sides, we get

$$\frac{y^2}{a_2^2} + \frac{x^2}{a_1^2}\cos^2\delta - \frac{2xy}{a_1a_2}\cos\delta = \left(1 - \frac{x^2}{a_1^2}\right)\sin^2\delta$$

or,

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1a_2}\cos\delta = \sin^2\delta \quad ...(6.11)$$

Equation (6.11) represents the general equation for an ellipse.

This generalized ellipse has been graphically represented in Fig. 6.12 (for $\delta \neq 0$ or $\delta \neq n\pi$ or $\delta \neq (2n+1)\frac{\pi}{2}$ and $a_1 \neq a_2$).

If $a_1 = a_2 = a$ (say), Eq. (6.11) reduces to

$$x^2 + y^2 - 2xy\cos\delta = a^2\sin\delta \quad ....(6.12)$$

This equation represents a general circle. This circle also can have its centre any where in the plane as the ellipse of Fig. 6.12.

Let us now have a look at the various cases of the general Eq. (6.11).

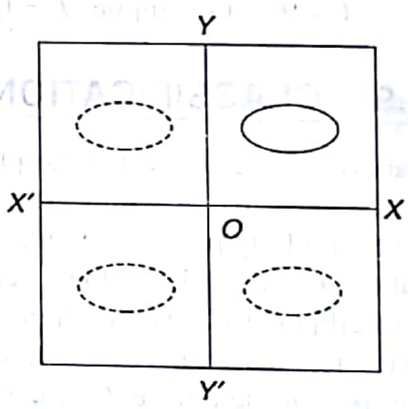


Fig. 6.12 Graphical representation of the general ellipse. It can remain in any of the four quadrants of the plane depending on the values of x and y

Case 1 When phase difference $\delta = 0, 2\pi, 4\pi, ...$ i.e., $\delta = 2n\pi$, for $n = 0, 1, 2, ...$, we obtain $\sin\delta = 0$ and $\cos\delta = 1$.

So, from Eq. (6.11), we get

$$\left(\frac{x}{a_1}-\frac{y}{a_2}\right)^2=0 \Rightarrow \frac{x}{a_1}-\frac{y}{a_2}=0$$

or, $$y=\frac{a_2}{a_1}x \qquad ...(6.13)$$

Equation (6.13) is an equation of a straight line passing through the origin with a slope $s=\frac{a_2}{a_1}\geq 0$. So, the resultant emergent light will be plane or linearly polarized with vibrations in the same plane as the two superposing components (Fig. 6.13).

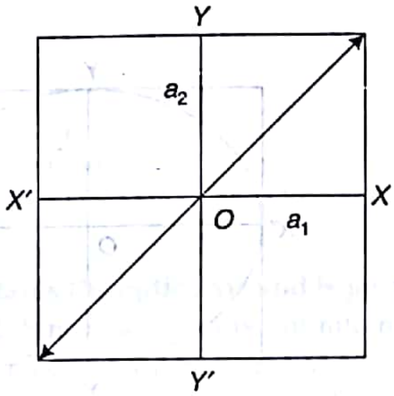


Fig. 6.13 Superposition of two plane polarized light rays to generate plane polarized light.

Case 2 When phase difference $\delta=\pi, 3\pi, 5\pi, ...,$ i.e., $\delta=(2n+1)\frac{\pi}{2}$, for $n=0, 1, 2, ...,$ we obtain $\sin\delta=0$ and $\cos\delta=-1$.

So, from Eq. (6.11), we get

$$\left(\frac{x}{a_1}+\frac{y}{a_2}\right)^2=0 \Rightarrow \frac{x}{a_1}+\frac{y}{a_1}=0$$

or, $$y=-\frac{a_2}{a_1}x \qquad ...(6.14)$$

Equation (6.14) is again that of a straight line passing through the origin with a slope $s=-\frac{a_2}{a_1}\leq 0$.

In this case also, the resultant emergent light will be plane polarized with vibrations as in Case 1 (Fig. 6.14).

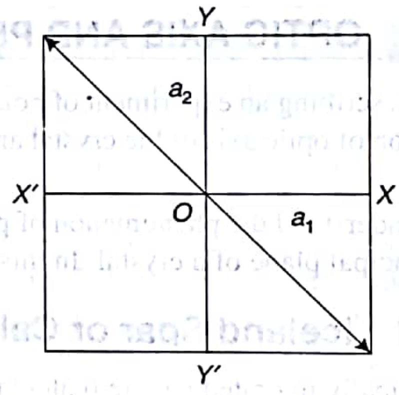


Fig. 6.14 Superposition of two plane polarized light rays to generate plane polarized light.

Case 3 When phase difference $\delta=\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...$ i.e., $\delta=(2n+1)\frac{\pi}{2}$, for $n=0, 1, 2, ...,$ $\sin\delta=1$ and $\cos\delta=0$. So from Eq. (6.11), we get

$$\frac{x^2}{a_1^2}+\frac{y^2}{a_2^2}=1 \qquad ... (6.15)$$

This equation (Eqn. (6.15)) is that of a symmetrical ellipse with major axis and minor axis coinciding with X-axis and Y-axis respectively. The emergent light will be elliptically polarized (Fig. 6.15).

Case 4 When phase difference $\delta=\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...,$ i.e., $\delta=(2n+1)\frac{\pi}{2}$ for $n=0, 1, 2, ...,$ and $a_1=a_2=a$ (say).

In this case Eq. (6.11) takes the following form,

$$x^2+y^2=a^2 \qquad ... (6.16)$$

The equation (Eq. (6.16)) represents a circle having the centre at origin. So, in this case the emergent light will be circularly polarized (Fig. 6.16).

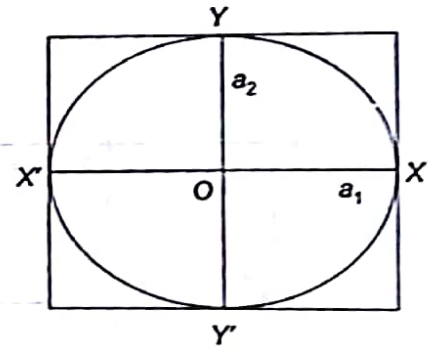


Fig. 6.15 Superposition of the plane polarized light ray to generate elliptically polarized light.

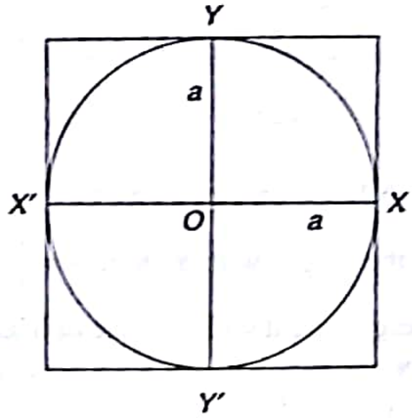


Fig. 6.16 Circularly polarized light formed by superposition of two linearly polarized light.

6.11 OPTIC AXIS AND PRINCIPAL PLANE OF A CRYSTAL

While describing an experiment of polarization with a tourmaline crystal in the Sec. 6.4, we had simply made a mention of optic axis of the crystal and we have also discussed optic axis in subsection 6.6.3 from one point of view.

To understand the phenomenon of polarization well, one has to acquire good idea regarding the optic axis and principal plane of a crystal. In this section we are discussing optic axis from other point of view.

6.11.1 Iceland Spar or Calcite Crystal

A chemically hydrated calcite (calcium carbonate, ($CaCO_3$)) crystal is transparent and colourless. It is also known as iceland spar. When a transparent variety of calcite is struck, it breaks obliquely in three different planes. Thus it breakes into a number of pieces where each piece takes a regular geometrical form of a rhombohedron like the one given in Fig. 6.17.

The six faces of the rhombohedron are parallelograms, having two acute angles of 78°5′ each and two obtuse angles of 101°55′ each. There are two diagonally opposite corners A and C' where the three obtuse angles (each of 101°55′) meet together and these two corners (A and C') are known as blunt corners. At each of the other six corners, two acute angles and one obtuse angle meet together. This crystal has the property of producing a pair of refrected rays for a single incident ray. For this reason, this crystal is known as double-refracting crystal.

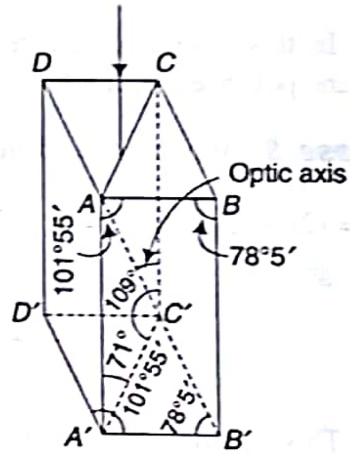


Fig. 6.17 Optic axis AC' is shown in the iceland spar (calcite crystal).

6.11.2 Optic Axis of a Crystal

A line passing through either of the two blunt corners (A or C') which is also equally inclined to the three faces that meet together at the corner gives the direction of the optic axis of the crystal, **the optic axis of a crystal is merely a direction and not any particular line**; so any other line which is parallel to the one mentioned above is also the optic axis of the crystal.

The special features of the optic axis are as follows:

(a) It is a direction rather than any particular line.

(b) The optic axis divides the crytal symmetrically.

(c) The velocities of both *O*-ray and *E*-ray along the optic axis are equal.

6.11.3 Principal Section of a Crystal

The principal section of a crystal is its section or plane which contains the optic axis and is perpendicular to the two opposite refracting faces or surfaces of the crystal. Since there can be infinite number of lines parallel to the direction of the optic axis, there can be infinite number of principal sections. In Fig. 6.17 one such principal section is $ACC'A'$. It has cut the crystal surface in a parallelogram ($ACC'A'$) with angle 71° ($\angle AA'C'$) and 109° ($\angle A'C'C$). In fact, each principal section cuts the crystal surface in a parallelogram having same angles.

6.12 NICOL PRISM

It is an ingenious optical device which is designed from a calcite crystal and can be used in optical instruments for producing and analyzing linearly polarized light. But recently it has been replaced by polaroids. While discussing double refraction (Section 6.6), we have observed that when a beam of ordinary (i.e., unpolarized) light is transmitted through a crystal of calcite, in general, two completely plane polarized beams, the beam of *O*-rays and the beam of *E*-rays, having vibrations in two mutually perpendiculer planes are obtained. Now if by some means we can eliminate one beam the calcite crystal may be used for obtaining plane polarized light from ordinary light. A device for elemination of the *O*-ray was accomplished in an ingeneous manner by William Nicol in 1828 by utilizing the phenomenon of total internal reflection of light at a thin film of canada balsam separating two pieces of calcite. This device is now known after his name as Nicol prism. But it is not really a prism, in fact it is a parallelo-pipendron.

6.12.1 Construction of Nicol Prism

A calcite crystal is shaped as rhombohedron from a natural crystal (having a length to breadth ratio of 3:1) by stricking. A cross-section $ABCD$ of the rhombohedron is shown in Fig. 6.18. Next, the end-faces containing AB and CD are properly ground by a grinding machine to make the angles in the principal section 68° and 112° in place of 71° and 109° respectively as a result of grinding the end faces now contain edges $A'B$ and CD'. The crystal is then cut into two halves along the plane which contain $A'D'$ the two surfaces are optically flat. Then they are cemented together by canada balsam — a transparent liquid with refractive index $\mu = 1.55$ for sodium light (wavelength: 589.3 nm). The crystal is finally enclosed in tube which is blackened inside and the construction of Nicol prism is then complete.

6.12.2 Action of Nicol Prism as Polarizer and Analyzer

(a) Polarizer Let a light ray PM of ordinary (unpolarized) light, almost parallel to BD', be incident on the face $A'B$. It splits into two refracted rays inside the crystal, MR and MQ — the so-called ordinary ray (*O*-ray) and extraordinary ray (*E*-ray) respectively. Both of the rays are plane or linearly polarized. The *O*-ray has vibrations perpendicular to the principal section of the crystal and the *E*-ray has vibrations in the principal section (i.e., vibrations parallel to the principal section). The refractive index μ_c of canada balsam lies between μ_e and μ_o, respectively the refractive indices of calcite crystal (the material of Nicol prism) for *E*-ray and *O*-ray, i.e., $\mu_e < \mu_c < \mu_o$.

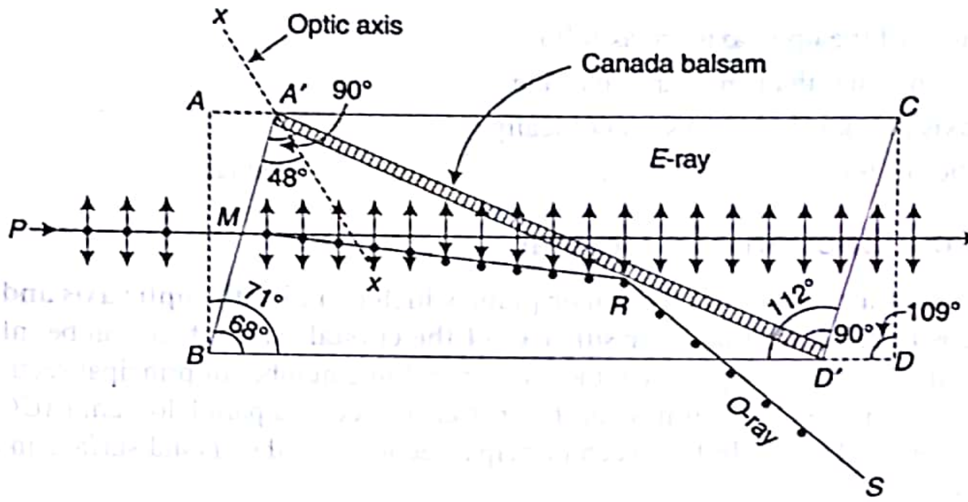


Fig. 6.18 The cross-section view of a Nicol prism

So, when the O-ray reaches the layer of canada balsam, it passes from an optically denser to an optically rarer medium. Since the crystal length is large, the O-ray is incident at the calcite canada balsam surface of separation at an angle greater than its critical angle (69°) and thus gets totally internally reflected and finally it gets absorbed by the blackened tube which encloses the crystal.

The E-ray, however, is transmitted by the calcite-canada balsam surface of separation and emerges from the Nicol prism as plane polarized light with vibrations parallel to the principal section of the crystal.

(b) Analyzer When an unpolarized ray of light is incident on a Nicol prism, the emergent ray becomes linearly polarized having vibrations in the principal section of the Nicol prism [Fig. 6.19(a)]. When the emergent light ray falls on a second Nicol prism whose principal section is parallel to that of first one, the vibrations remain on the principal section of the second one and the light ray, behaving as E-ray, is transmitted through it completely. The intensity of the emergent light ray becomes maximum.

If now the second Nicol prism is rotated to make its principal section normal to that of the first one (Fig. 19(b)], the vibrations of the incident polarized light becomes normal to its principal section and the ray behaves as O-ray inside the second Nicol prism. So, it is lost by total internal reflection at the calcite canada balsam surface of separation. In this situation, no light ray gets imerged from the second Nicol prism and the two Nicol prisms P and A are said to be crossed to each other.

If the second Nicol prism (A) is further rotated tilll its principal section is again parallel to that of the first Nicol prism (P) [Fig. 19(c)], the intensity of the emergent light again becomes maximum. By using a rotating Nicol prism, we can detect whether the incident light ray is linearly (plane) polarized or not. If the intensity of the emergent light varies between a maximum and a minimum and the value of the minimum is zero, then the incident light ray is plane polarized.

Here, the Nicol prism P acts as polarizer and A acts an analyzer. Whenever we need to use polarizer and analyzer of light beams for various purposes, we can use Nicol prisms. There is a limitation of the use of Nicol prism, it can be used only when the incident beam is slightly divergent or convergent. It cannot be used in a situation where the incident beam of light is highly divergent or convergent.

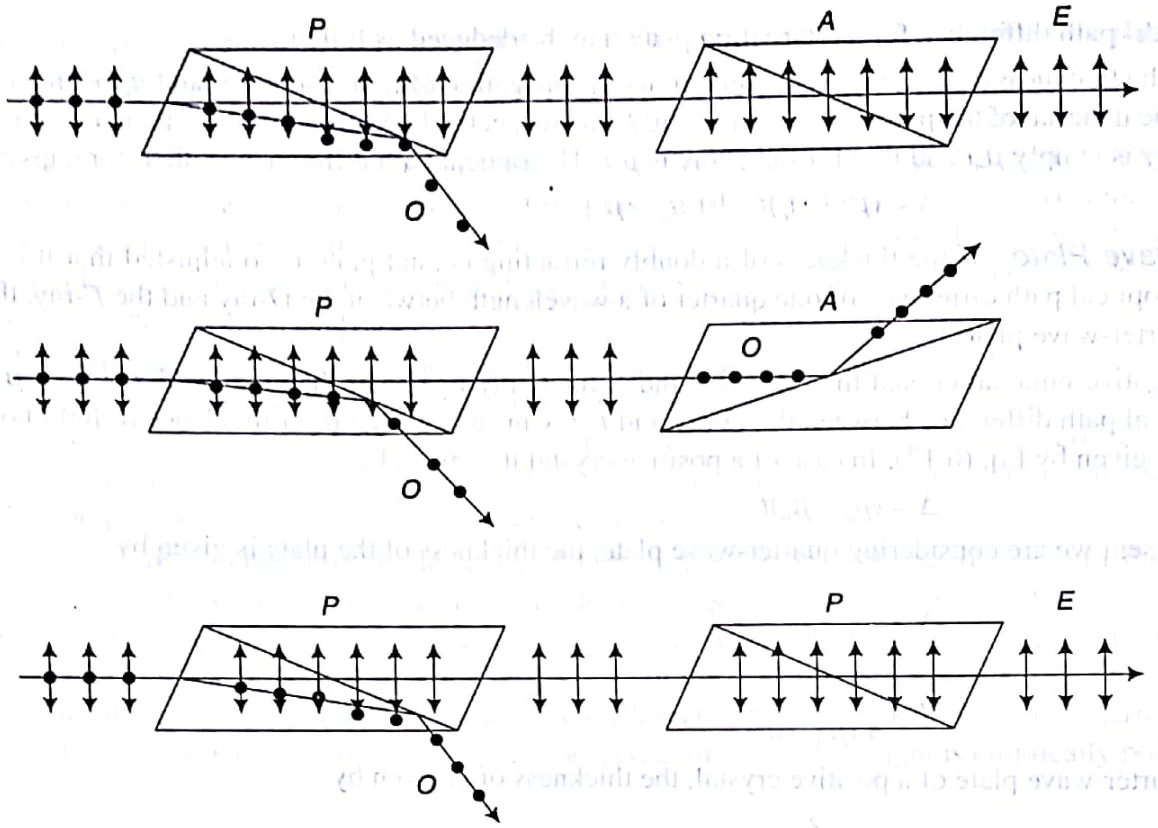


Fig. 6.19 Use of two Nicol prisms; one as polarizer (p) and the other as analyzer (A)

6.13 RETARDATION PLATES (QUARTER OR HALF-WAVE PLATES)

Definition A retardation plate is an optical device which can make a finite optical path difference between the O-ray and the E-ray while they travel through it in the same direction by retarding the motion of one of these rays.

A plate cut from a doubly refracting crystal by making sections parallel to the optic axis can be used as a retardation plate (see subsection 6.6.3).

If both the rays (O-ray and E-ray) propagate along the optic axis, they have same velocities and they have different velocities in any other direction. If the rays propagate along a direction normal to the optic axis of the plate, the difference in the velocities of O-ray (v_o) and E-ray (v_e) is maximum. So by allowing light to prop agate through a direction normal to the optic axis a plate of calcite or similar crystal cut parallel to the optic axis can be used as a retarding plate. Usually, there are two types of retardation plates in use. They are quarter-wave plates and half-wave plates (they are very frequently denoted by $\frac{\lambda}{4}$ - plate and $\frac{\lambda}{2}$ - plate respectively).

The optical path difference for a retardation plate may be deduced as follows:

Let t be the thickness of the plate in the direction of propagation of light and let μ_o and μ_e be the refractive indices of the material of the plate for the O-ray and E-ray respectively. Within the plate, then, the optical path for the O-ray is simply $\mu_o t$ and that for the E-ray is $\mu_e t$. The optical path difference is, therefore, given by

$$\Delta = (\mu_o - \mu_e)t \quad [\text{if } \mu_o > \mu_e] \qquad ...(6.17)$$

Quarter-wave Plate If the thickness of a doubly refracting crystal plate is so adjusted that it is able to produce an optical path difference of one quarter of a wavelength between the O-ray and the E-ray, then it is called a quarter-wave plate.

For a negative uniaxial crystal the refractive index for O-ray is greater than that of E-ray (i.e., $\mu_o > \mu_e$). So, the optical path difference between the O-ray and E-ray in such a crystal for incidence of light normal to optic axis is given by Eq. (6.17). In case of a positive crystal it is given by

$$\Delta = (\mu_e - \mu_o)t \qquad ...(6.18)$$

As at present we are considering quarter-wave plate, the thickness of the plate is given by

$$\Delta = (\mu_o - \mu_e)t = \frac{\lambda}{4}$$

or,

$$t = \frac{\lambda}{4\,(\mu_o - \mu_e)} \qquad ...(6.19)$$

For a quarter-wave plate of a positive crystal, the thickness of is given by

$$t = \frac{\lambda}{4\,(\mu_e - \mu_o)} \qquad ...(6.20)$$

A quarter-wave plate is used to produce circularly and elliptically polarized light by placing it in the path of a plane polarized light. Further, in conjunction with a Nicol prism, it becomes a versatile analyzer and can analyze all kinds of polarized light.

Half-wave Plate If the thickness of a doubly refracting crystal plate is so adjusted that it is able to produce an optical path difference of one half of a wavelength between the O-ray and the E-ray, then it is called a half wave plate.

For a negative crystal like calcite ($\mu_o > \mu_e$), the path difference for a half-wave plate is given by

$$\Delta = (\mu_o - \mu_e)t = \frac{\lambda}{2} \quad [\text{by Eq. (6.17)}]$$

So the thickness,

$$t = \frac{\lambda}{2\,(\mu_o - \mu_e)} \qquad ...(6.21)$$

For a positive crystal plate ($\mu_e > \mu_o$)

The thickness,

$$t = \frac{\lambda}{2\,(\mu_e - \mu_o)} \qquad ...(6.22)$$

6.14 DETECTION OF POLARIZED, PARTIALLY POLARIZED AND UNPOLARIZED LIGHT

According to Stokes, light can be classified into seven classes on the basis of its status of polarization. These classes are (i) unpolarized, (ii) linearly polarized, (iii) circularly polarized, (iv) elliptically polarized,

(v) mixture of unpolarized and linearly polarized, (vi) mixture of unpolarized and circularly polarized and (vii) mixture of unpolarized and elliptically polarized light.

In order to detect the status of polarization of a given beam of light, the beam at first may be allowed to pass through an analyzer which is capable of rotating about the direction of propagation of the light beam as axis [Fig. 6.20 (a)]. This can be considered as the first step of analysis. If the status of polarization of the light beam is not completely known in this step, then one can carry on a second test with the help of a pair of one quarter-wave plate and one analyzer separated by a distance as shown in Fig. 6.20(b). This may be considered as second step of the detection analysis.

If, in the first step, no variation of intensity of the transmitted light beam is observed, then it may be either unpolarized or circularly polarized beam of light. If two maxima and two minima are observed in the first step, then the given beam of light may be either partially polarized or elliptically polarized. And if two maxima and two extinctions are observed in the first step, then there is no need to carry out the second step of the detection process. The given light beam is linearly polarized.

On observation of no change in the intensity of light in the first step, the second step of the detection process is carried out. In this step if no change of intensity is observed, then the given light is unpolarized. If two maxima and two extinctions are observed, then it is circularly polarized. Again, on observation of two maxima and two minima in the first step, the second step of the detection process is carried out. In this step if two maxima and two minima of intensity are observed, then the given light is partially polarized. On the other hand, if two maxima and two extinctions are observed, then the given light is elliptically polarized.

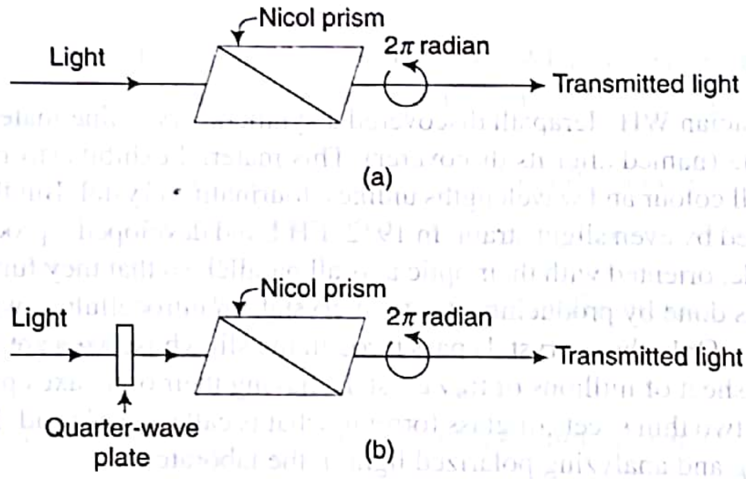


Fig. 6.20 Detection of unpolarized and polarized light in two steps.

6.15 DICHROIC CRYSTALS AND POLAROIDS

We have discussed various methods of generating plane polarized beams of light in Section 6.6. Selective absorption was one of them. In double refraction method of polarization, we came across the phenomenon of selective absorption. In this section we would like to throw more light on this matter.

6.15.1 Dichroic Crystals

Some doubly refracting minerals and crystals show the property of absorption of one of the two doubly refracting rays (i.e., *O*-ray and *E*-ray) very strongly by allowing the other to transmit through them with very little loss. **This phenomenon of selective absorption of light rays is known as dichroism and those crystals which exibit the property of dichroism are called dichroic crystals.**

Tourmaline is a dichroic crystal and absorbs the ordinary ray (O-ray) completely as shown in Fig. 6.21. But the extra-ordinary ray (E-ray) is partly absorbed and so it emerges out of the crystal. Thus if an unpolarized light is allowed to pass through a dichroic crystal of proper thickness, one of the components is totally absorbed and then other component transmits in appreciable amount. The emergent beam will thus be plane polarized. A thin tourmaline crystal of approximately 1 mm thickness with its faces cut parallel to its optic axis can be used as a polarizer. But as the polarized light of a tourmaline crystal is coloured due to unequal absorption of the light of various wavelengths, its use as a polarizer is limited.

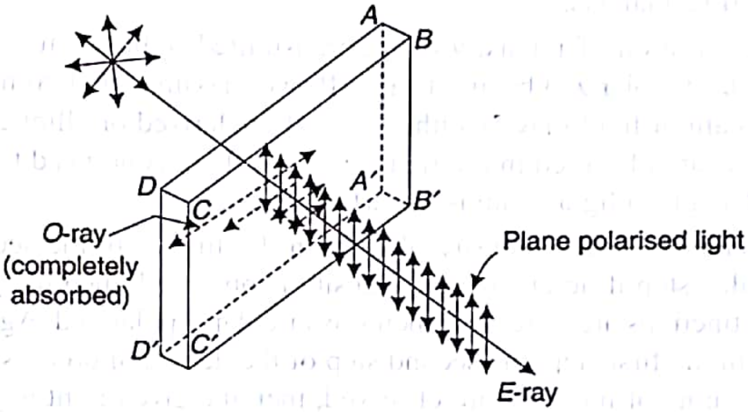


Fig. 6.21 Dichroism exhibited by a crystal. It absorbs the O-ray completely and allows E-ray to pass through.

6.15.2 Polaroids

In 1852, an English Physician WH Herapath discovered a synthetic crystaline material iodo-sulphate of quinine known as herapathite (named after its discoverer). This material exhibits strong dichroism. It transmits plane polarized light of all colour and wavelengths unlike a tourmaline crystal. But these crystals are not at all stable and they are affected by even slight strain. In 1932, EH Land developed a process which arranges herapathite crystal side by side, oriented with their optic axis all parallel, so that they function like a single crystal of big dimensions. This is done by producing a paste of crystals in nitrocellulose which is then squeezed out through a very narrow slit. Only those crystals pass through the slit whose axes are parallel to the length of it thereby producing a fine sheet of millions of tiny crystals, having their optic axes parallel to each other. This is then mounted between two thin sheets of glass forming what is called a polaroid. It has largely replaced the Nicol prism for producing and analyzing polarized light in the laboratory.

Another variety of long-sized polaroids called H-polaroids is formed by heating and stretching polyvinyl alcohol films so as to orient the complex molecules in the direction of stress, which then become double refracting. And when these polaroids are impregnated with iodin, they exhibit the characterization of dichroism. These polaroids are colourless and transmit 33% more light than the herapathite polaroids.

It was later discovered by Land and Rogers that when stretched polyvinyl alcohol film is heated along with a dehydrating catalyst, e.g., HCl, it darkens slightly but shows strong dichroism and becomes very stable. This polaroid is called K-polaroid. When two pieces of polaroids are crossed, light is perfectly extinguished but if they are made parallel to each other, the light transmitted by the first one is also transmitted by the second one [Fig. 6.22(a) and (b)].

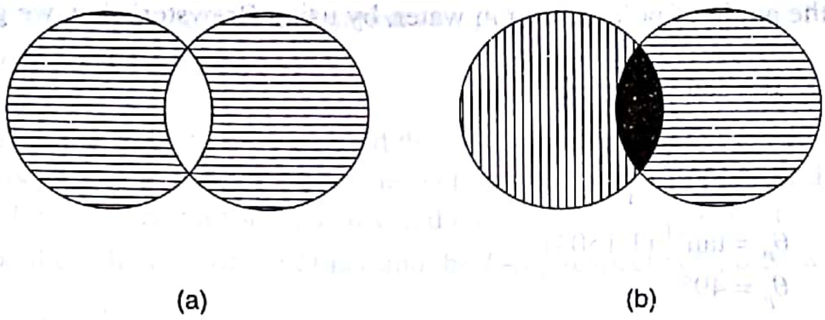


Fig. 6.22 (a) Two parallel polaroids transmiting light (b) Two crossed polaroids showing extinction of light

Applications of polaroids are *versatile*. They are cheaper than the Nicol prisms and are used in laboratories for producing and analyzing of linearly polarized light. Polaroids are also used in the sunglasses for cutting off glare of light coming from horizontal surfaces.

To cut off the dazzling light of an approaching car which is coming from opposite directions, *K*-polaroids are extensively used in the headlights and the windscreens of cars. They are also used in windows of trains and aeroplanes to control intensity of light entering into them and in viewing stereoscopic motion pictures.

Worked-out Examples

Example 6.1 The angle of polarization of a material is 60°. What is the (i) refractive index of the transparent material, (ii) the angle of refraction corresponding to the angle of incidence, θ_i (= 60°), and (iii) the angle between the reflected and refracted rays?

Sol. (i) Here, $\theta_p = 60°$. From Brewster's law, we get, $\mu = \tan\theta_p = \tan 60° = 1.73$.

(ii) Also from Brewster's law, we know that for incidence at polarization angle, the sum of angle of reflection and that of refraction is 90°.

$$\therefore \quad \theta_p + \theta_r = 90° \quad [\because \text{angle of reflection } \theta_{rl} = \text{polarizing angle} = \theta_p]$$

$$\therefore \text{ angle of refraction} \quad \theta_r = 90° - \theta_p = 90° - 60° = 30°$$

(iii) Again, since $\theta_p + \theta_r = 90°$, the sum of angle of reflection (θ_{rl}) and angle of reflection (θ_r) is given by

$$\theta_{rl} + \theta_r = \theta_p + \theta_r = 90° \quad [\because \theta_{rl} = \theta_i = \theta_p]$$

$\therefore$ the angle between the reflected and the refracted rays is 90°

Example 6.2 Calculate the polarizing angle for a light ray travelling from water of refractive index 1.33 to glass of refractive index 1.53. [WBUT 2008]

Sol. The refractive index of water with respect to air is ${}_a\mu_w = 1.33$ and that of glass with respect to air is ${}_a\mu_g = 1.53$.

So, the refractive index of glass with respect to water is given by

$$\mu = {}_w\mu_g = \frac{{}_a\mu_g}{{}_a\mu_w} = \frac{1.53}{1.33}$$

Now, if θ_p be the angle of polarization in water, by using Brewster's law we get

$$\mu = \tan\theta_p = \frac{1.53}{1.33}$$

or, $$\theta_p = \tan^{-1}\left(\frac{1.53}{1.33}\right)$$

or, $$\theta_p = \tan^{-1}(1.1504)$$

or, $$\theta_p = 49°$$

Example 6.3 Two polarizers are crossed to each other, a third polarizer is placed between them which makes angle of θ with the first polarizer. An unpolarized light of intensity I_o is incident on the first one and passes through all of the three polarizers. Calculate the intensity of the light which emerges from the second polarizer. [WBUT 2008]

Sol. The angle between the first and third polarizer is θ and that between the third and second polarizer is

$$\theta' = 90° - \theta.$$

So, by using Malus law, the intensity of the light emerged from the third polarizer is given by

$$I' = I_o \cos^2\theta$$

Similarly, the intensity of the light emerged from the second polarizer is given by

$$I = I' \cos^2\theta'$$

or, $$I = I_o \cos^2\theta \cos^2(90° - \theta)$$

or, $$I = I_o \sin^2\theta \cos^2\theta$$

$\therefore$ the light which finally emerges from the second polarizer has an intensity of $I = I_o \sin^2\theta \cos^2\theta$.

Example 6.4 What is the refractive index of glass if the light of 550 nm wavelength is completely plane polarized when reflected at an angle of 60°?

Sol. As the reflected light is completely plane polarized, the angle of incidence is the angle of polarization for glass. So, from Brewster's law we get the required refractive index μ as follows

$$\mu = \tan\theta_p = \tan 60° = \sqrt{3}$$

$\therefore$ $$\mu = 1.732$$

Example 6.5 The refractive indices of canada balsam and calcite crystal are 1.550 and 1.658 respectively for the ordinary ray (i.e., O-ray) formed through double refraction. Find out the maximum angle of inclination of the O-ray with the surface of canada balsam so that it is still quenched.

Sol. The refractive indices of canada balsam and calcite crystal are $\mu_b = 1.550$ and $\mu_c = 1.658$. Let the critical angle of the O-ray in the crystal of calcite be θ_c.

$\therefore$ $$\sin\theta_c = \frac{\mu_b}{\mu_c} = \frac{1.550}{1.658} = 0.935.$$

Hence, $$\theta_c = 69.2°$$

So, when the O-ray is incident on the surface of canada balsam at an angle of inclination $\theta \le (90° - \theta_c)$ the O-ray will quench.

∴ the required maximum angle of inclination (θ_{max}) for quenching is given by

$$\theta_{max} = 90° - 69.2° = 20.8°$$

Example 6.6 Calculate the velocity of O-ray and that of E-ray in a calcite crystal in a plane which is normal to the optic axis. Also, find the ratio of the greater velocity to the smaller one and comment on the ratio. The refractive indices of calcite crystal for the E-ray and the O-ray are 1.485 and 1.659 respectively.

Sol. Let v_o and v_e be the velocities of the O-ray and the E-ray respectively, then we get,

$$\mu_o = \frac{c}{v_o} \quad \text{and} \quad \mu_e = \frac{c}{v_e}$$

where c is the velocity of light in the free space.

$$\therefore \qquad v_o = \frac{c}{\mu_o} = \frac{3 \times 10^8}{1.659} = 1.808 \times 10^8 \text{ m/s}$$

and

$$v_e = \frac{c}{\mu_e} = \frac{3 \times 10^8}{1.485} = 2.020 \times 10^8 \text{ m/s}$$

∴ the velocity of E-ray is greater than that of O-ray and the ratio is given by

$$r = \frac{v_e}{v_o} = \frac{2.020}{1.808} = 1.12$$

∴ the E-ray moves faster than the O-ray and its velocity is 1.12 times of the velocity of the O-ray.

Example 6.7 Calculate the thickness of a mica sheet required to make a quarter wave-plate for $\lambda = 546$ nm. The refractive indices for the O-ray and E-ray in mica are 1.586 and 1.592 respectively. [WBUT 2002]

Sol. From the given data, we find that $\mu_e = 1.592$ and $\mu_o = 1.586$ for mica and $\mu_e > \mu_o$. So, mica is positive crystal and for this crystal, the required thickness is given by

$$t = \frac{\lambda}{4\,(\mu_e - \mu_o)}$$

$$\therefore \qquad t = \frac{546 \times 10^{-9}}{4\,(1.592 - 1.586)} \text{ m}$$

or,

$$t = \frac{546 \times 10^{-9}}{4 \times 0.006} \text{ m}$$

or,

$$t = 22.75 \times 10^{-6} \text{ m}$$

or,

$$t = 22.75\ \mu\text{m}$$

Example 6.8 A sheet of cellophane is a half-wave plate for light of $\lambda = 4 \times 10^{-5}$ cm. Assuming that there is negligible variation in indices of refraction with wavelength λ, how would the sheet behave with respect to the wavelength of $\lambda' = 8 \times 10^{-5}$ cm ?

Sol. Let t be the thickness of the half-wave plate, so we can write,

$$t = \frac{\lambda}{2\,(\mu_e - \mu_o)} = \frac{4 \times 10^{-5}}{2\,(\mu_e - \mu_o)}$$

or,

$$t = \frac{8 \times 10^{-5}}{4\,(\mu_e - \mu_o)} = \frac{\lambda'}{4\,(\mu_e - \mu_o)}$$

which is the formula for a quater wave plate for wavelength $\lambda' = 8 \times 10^{-5}$ cm.

So, for $\lambda' = 8 \times 10^{-5}$ cm wave, the sheet would behave as a quarter-wave plate.

Example 6.9 A quartz plate is a half-wave plate for light whose wavelength is λ. Assuming that the variations in the indices of refraction with wavelength can be neglected, how would this behave with respect to light of wavelength $\lambda_1 = 2\lambda$?

Sol. The thickness of half-wave plate is given by

$$t = \frac{\lambda}{2(\mu_o - \mu_e)}$$

which can be easily re-written as follows:

$$t = \frac{2\lambda}{4(\mu_o - \mu_e)} = \frac{\lambda_1}{4(\mu_o - \mu_e)}$$

which is the expression for a quarter-wave plate for a light with wavelength λ_1.

So, the given plate will behave as a quarter wave plate for light of wavelength λ_1.

Review Exercises

Part 1: Multiple Choice Questions

1. Which of the following phenomena causes polarization of light?
   (a) Reflection (b) Refraction (c) Double refraction (d) Diffraction
2. Linearly polarized light can be produced by
   (a) simple reflection (b) simple refraction
   (c) reflection at the angle of polarization (d) none of the above
3. Which of the following phenomena proves that light is a transverse wave?
   (a) Interference (b) Dispersion (c) Diffraction (d) Polarization
4. Plane polarized light can be produced by
   (a) reflection at polarizing angle (b) Nicol prism
   (c) piles of plates (d) all of the above
5. In case of ellipically polarized light which of the following statements is true?
   (a) Amplitude of vibrations changes in direction only
   (b) Amplitude of vibration changes in magnitude only
   (c) Amplitude of vibration changes both in magnitude and direction
   (d) None of the above three
6. From polarization of light, one can conclude that
   (a) light is a transverse wave
   (b) light is a longitudinal wave
   (c) light can bend while facing a sharp edge of an object
   (d) none of the above

7. The plane of vibration makes an angle θ with that of polarization. The value of θ is
   (a) 0° (b) 90°
   (c) 45° (d) none of the above
8. An unpolarized light consists of
   (a) infinite number of plane polarized light (b) finite number of plane polarized light
   (c) only two plane polarized light (d) none of the above
9. In plane polarized light
   (a) the magnitude of light vectors changes and the orientation remains same
   (b) the magnitude of light vectors remain same and the orientation changes
   (c) both of magnitude and orientation change continuously
   (d) none of the above three
10. Number of optic axes in a uniaxial crystal is
    (a) one (b) two (c) five (d) ten
11. Number of optic axes in a bi-axial crystal is
    (a) one (b) five (c) two (d) ten
12. The optic axis is a direction along which
    (a) the *O*-ray travels faster than the *E*-ray
    (b) the *E*-ray travels faster than the *O*-ray
    (c) both *O*-ray and *E*-ray travel with the same velocity
    (d) none of the above
13. In a quarter-wave plate, the path difference (in terms of wavelength λ) between the *O*-ray and the *E*-ray is
    (a) $\frac{\lambda}{4}$ (b) zero
    (c) $\frac{\lambda}{2}$ (d) none of the above
14. In a half-wave plate, the phase difference between the *O*-ray and the *E*-ray is
    (a) $\frac{\pi}{2}$ (b) π
    (c) zero (d) none of the above
15. If θ_p be the angle of polarization, then the refractive index μ of the material is given by
    (a) $\sin\theta_p$ (b) $\cos\theta_p$ (c) $\tan\theta_p$ (d) $\sec\theta_p$
16. A Nicol prism can act as a
    (a) polarizer (b) analyzer
    (c) both analyzer and polarizer (d) none of the above
17. If the refractive index of water is 1.33, the polarizing angle of light reflected from the surface of a pond is given by
    (a) $\cos^{-1}(1.33)$ (b) $\tan^{-1}(1.33)$ (c) $\cot^{-1}(1.33)$ (d) $\sin^{-1}(1.33)$

18. An elliptically polarized light is a general case of
    (a) only the linearly polarized light
    (b) only the circularly polarized light
    (c) both of linearly and circularly polarized light
    (d) none of the above
19. If light is incident at the angle of polarization then the angle between the refelcted ray and the refracted ray is
    (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) π (d) $\frac{3\pi}{2}$
20. The action of Nicol prism is based on the phenomenon of
    (a) scattering (b) double refraction (c) refraction (d) reflection

Answers

1. (c)	2. (c)	3. (d)	4. (d)	5. (c)	6. (a)	7. (b)	8. (a)
9. (a)	10. (a)	11. (c)	12. (c)	13. (a)	14. (b)	15. (c)	16. (c)
17. (b)	18. (c)	19. (a)	20. (b)				

Short Questions with Answers

1. **Why is polarization so important in case of light?**
   The phenomenon of polarization is very important in case of light, because it is polarization which is capable of proving that light is a transverse wave. No other property of light can confirm this fact.
2. **What is polarization? State Brewster's law of polarization?** [WBUT 2002, 2004]
   Polarization is a phenomenon of light due to which the vibrations of the electric and magnetic vectors of a light wave can be restricted to remain confined in a plane only.
   Brewster's law states that at angle of polarization, the refractive index of a polarizing substance is equal to the tangent of the angle of polarization and the angle of polarization is that incident angle for which the reflected beam becomes completely polarized. Also, the angle between the reflected ray and the refracted ray becomes $\frac{\pi}{2}$ radian.
3. **Explain why light waves can be polarized but that of sound cannot be polarized.**
   As we know, the sound waves are longitudinal waves where the particles of the medium vibrate along the direction of propagation of the wave, for this reason, they cannot be polarized. The propagation of the wave cannot be cut-off due to the longitudinal nature of it. But in case of light waves, polarization is possible as it is a transverse wave. The polarization of light proves that light is a transverse wave.
4. **What is the difference between the natures of ordinary and extra-ordinary rays of light when they pass through positive and negative crystals?**
   In case of the positive crystal, the ordinary ray moves faster than the extraordinary ray and hence the refractive index of the extraordinary ray is more than that of ordinary ray. Quatz crystal is an example of positive crystal. On the other hand, in case of a negative crystal, the extraordinary ray moves faster than the ordinary ray, and hence the refractive index of the ordinary ray is greater than the extraordinary ray. Calcite crystal is an example of negative crystal.

5. What are meant by uniaxial and biaxial crystals? Cite one example of each one.

In some crystals there exists only one direction (or optic axis) along which the velocities of ordinary and extraordinary rays are same. Such crystals are called uniaxial crystals. There is another kind of crystal in which there exist two optic axes along which the velocities of O-ray and E-ray are same. These type of crystals are called biaxial crystals. Examples of uniaxial crystals are calcite and quartz crystals. Examples of biaxial crystals are aragonite and copper sulphate crystals.

6. Show that the maximum transmission through any polarizer is half of the maximum intensity of the incident light.

Let I_o be the intensity of the light which is incident on the polarizer. Let θ be the angle made by the direction of polarization of the polarizer and that of vibrations of the electric vector in the incident light beam. So, according to Malus law, the intensity of the light which has been transmitted through the polarizer is given by the following equation:

$$I = I_o \cos^2 \theta$$

where I is the intensity of the polarized light.

The incident light beam is unpolarized one. So, it vibrates in all directions on the plane normal to the direction of propogation of the light. So, in order to compute the intensity I, one has to take the time-average value of I. Now, the time average value of $\cos^2 \theta$ is 0.5. Hence, $I_{\max} = I_o \cos^2 \theta = 0.5\, I_o$. An ideal polarizer will be able to transmit this maximum quantity which is equal to one-half of the intensity of the incident light. This implies that even an ideal polarizer can at best have a maximum transmission which is only fifty percent of the light incident on the polarizer.

7. Two polarizers are placed at crossed position (angle between their polarizing planes is 90°) a third polarizer making an angle θ with the first one is placed between them. An unpolarized light of intensity I_o is incident on the first one and passes through all the three polarizers. Find the intensity of the emergent light. [WBUT 2008]

Let us use the Malus law in two steps. In the first step, the first and the third polarizers are considered the first polarizer acts as a polarizer and the third one making and angle θ with the first one acts as an analyzer.

So, by using Malus law, we can write

$$I_1 = I_o \cos^2 \theta$$

where I_1 is the intensity of the light which is coming out of the third polarizer.

Again, if we apply Malus Law considering the third polarizer as polarizer and second one as analyzer, we can write

$$I = I_1 \cos^2 (90° - \theta) = I_1 \sin^2 \theta$$

or, $$I = I_o \cos^2 \theta \sin^2 \theta$$

Hence, the intensity of the emergent ray is given by

$$I = \frac{I_o}{4} \sin^2 (2\theta)$$

8. Why is the colour of the sky blue?

It was proved by Lord Rayleigh that the percentage of scattered hight varies inversely as the fourth power of the wavelength of it when the size of the scattering particles is comparable to the wavelength of light waves. In case of blue light, the scattering is 16 times greater than that of red light. While the white light of the sun falls on the small particles in the atmosphere of the earth, the blue

light is scattered and polarized far more than the light of any other colour. For this reason, the sky appears to be blue. This blue colour of the sky is more brilliant when the atmosphere of the earth remains free from the dust particles and water molecules and only the scattering is due to the gas molecules.

9. Why does the sky appear to be red at the time of sunrise and sunset?

During sunset and sunrise, the sun remains near the horizon, so the light rays coming from the sun have to traverse a greater distance to reach the eyes of the observer as the surface of the earth is spherical the percentage of the dust particles (including smoke) are more near the earth's surface than the upper sky and as the size of the particles are comparable to the wavelength of the blue light, it gets scattered (and polarized) by these particles and lost. So the blue colour is removed from the sunlight in this process of scattering and the red colour with its largest wavelength remains more in the rest sunlight. At the time of the sunrise and sunset, such lights are reflected by the clouds and thus the colour of the sky appears to be red.

10. Discuss Nicol prism as polarizer and analyzer.

Refer to Article 6.12.2.

Part 2: Descriptive Questions

1. Explain the terms (a) polarization of light, (b) plane of polarization, (c) plane of vibration, and (d) optic axis.
2. Distinguish between polarized and unpolarized light. What is meant by principal section of a crystal?
3. Define polarization. How can you distinguish between circularly and elliplically polarized light? [WBUT 2002]
4. Write short notes on (a) positive and negative crystals, and (b) Nicol prism. [WBUT 2003]
5. Write a note on Nicol prism and its working as polarizer and analyzer. [WBUT 2004]
6. What is polarization of light? What do you mean by positive and negative crystals? How can you distinguish between an unpolarized, a circularly polarized and a plane polarized light. [WBUT 2004]
7. Write the name of different types of polarized lights.
8. Distinguish diffraction and double refraction. What are the advantages of a polaroid over a Nicol prism?
9. What is a Nicol prism? How is it prepared? What do you mean by crossed and parallel Nicol prisms?
10. State and explain Brewster's law of polarization by clearly indicating the nature of polarization of the reflected and the refracted rays.
11. What are the differences between (i) isotropic and anisotropic media, (ii) uniaxial and biaxial crystals, and (iii) positive and negative crystals?
12. What are *O*-ray and *E*-ray? How is the concept of total internal reflection used in the construction of a Nicol prism? Discribe the method of obtaining a polarized light with the help of a Nicol prism.
13. Describe the construction of a quarter-wave plate and explain how you can produce circularly and elliptically polarized light in such a plate.
14. How can you distinguish unpolarized linearly polarized, circularly polarized and elliptically polarized light?

Part 3: Numerical Problems

1. The critical angle of light in a certain substance is 45°. What is the polarizing angle?
2. Calculate the thickness of a quarter-wave plate for light of wavelength λ = 589.3 nm, given μ_o = 1.544 and μ_e = 1.553.
3. Plane polarized light passes through a quartz plate with its optic axis parallel to the faces. Calculate the least thickness of the plate for which the emergent beam will be plane polarized.
4. Calculate the thickness of a half-wave plate given that μ_e = 1.533 and μ_o = 1.544 and λ = 500 nm.
5. Calculate the thickness of a double refracting plate capable of producing a path difference of $\frac{\lambda}{4}$ between the E-ray and the O-ray.
6. A polarizer and an analyzer are oriented so that the amount of light transmitted is maximum. To what fraction of its maximum value is the intensity of the transmitted light reduced when the analyzer is rotated through an angle of (i) 30°, (ii) 45°, (iii) 60° and (iv) 90°?
7. The refractive index of a glass plate is 1.60. Calculate the angle of polarization and the corresponding angle of refraction.
8. An unpolarized light wave of intensity 10 mW/cm^2 passes through two Nicol prisms with principal section at 30° to each other. Calculate the intensity of the transmitted wave.
9. A beam of plane polarized light is changed into a circularly polarized light by passing it through a 0.003 cm thick slice of crystal. Calculate the difference in the indices of two rays in the crystal assuming it to be the minimum thickness that will produce the effect ($\lambda = 6 \times 10^{-7}$ m).

CHAPTER

7

Laser Optics

7.1 INTRODUCTION

Laser is an important discovery of modern physics and it gave the branch of optics a new impetus. The beam of light emitted from a laser source can have high degree of coherency, monochromaticity directionality and light-power density. All these properties of laser made it so important.

To be familiar with the history of laser one has to go through the history of development of maser. The word maser is an acronym of microwave amplification by stimulated emission of radiation while the word laser is that of light amplification by stimulated emission of radiation. Historically, the laser is nothing but outgrowth of the maser. In 1917, Einstein first theoretically predicted the stimulation process (i.e., stimulated emission) in addition to the spontaneous emission in a quantum system while establishing two relations known after him between two coefficients known as Einstein's A and B coefficients. Charles H Townes and his associates at the university of Columbia, USA and two Russian physicists Aleksander M Prokhorov and Nicolay G Basov independently successfully built maser in early fifties of the 20th century. The aforesaid trio were honoured with the Nobel prize in physics in 1964 for their outstanding work in quantum electronics which ultimately led to the discovery of maser and laser. In 1958, A H Shawlow and C H Townes set forth the principles of laser, then known as optical maser. And because of the development of laser optics in the sixties of last century, the attention of researchers in physics was drawn afresh in the field of optics. The first successful laser based on the principles of Schawlow and Townes was built in 1960.

Laser is one such gift of science that has found a good number of applications in the fields of biology medicine, chemistry, metallurgy and aeronautics. Townes and his co-workers built maser in 1954 by using ammonia (NH_3) gas working at the operation frequency of 2.4×10^{11} cycles/second. Schawlow and Townes in 1958 tried maser technique in the visible range of frequency. Finally T H Maiman could have successfully constructed the first optical maser (i.e., laser) in the year of 1960.

7.2 CHARACTERISTICS OF LASER

Laser has four distinct characteristics which were just mentioned in the previous section. Laser can be easily distinguished from ordinary light because of these four distinct chararistics of it though both of them belong to the same visible range of electromagnetic radiation. The said four striking features of laser are (a) high monochromaticity, (b) high degree of coherence, (c) high directionality (or low divergence), and (d) high brightness (or high power density).

(a) High Monochromaticity of Laser

The energy levels of the atoms which are responsible for generation of a laser beam through transitions of electrons from higher to lower levels are not completely discrete and sharp. A transition of an electron in an atom between two energy levels produce an emission or absorption of a photon whose wavelength lies within the range between λ and $\lambda + d\lambda$. This variation in the wavelength (or frequency) of radiation is called spectral broadening. Doppler broadening, collision broadening and natural broadening are three most important broadening mechanisms which are responsible for creation of spectral broadening. At the time of emitting a radiation, the atoms do not remain at rest. In fact, the frequency of the emitted radiation varies slightly depending on the velocities of the atoms concerned and their direction of motion. This is known as Doppler broadening. When an atomic electron in a solid substance causes emission of radiation through transition between two energy levels and produces a photon, an exponential decay of the amplitude of the wavetrain is observed. It is just like a collision broadening where shortening of the wave train occurs and results in broadening of the spectral line. It is called natural broadening.

When a source emits light the degree of monochromaticity of the light can be expressed in terms of line width or spectral width of the souce which is the frequency spread of the spectral line and can be denoted by $\Delta\nu$. We know that the frequency ν is given by

$$\nu = \frac{c}{\lambda}$$

or,

$$\Delta\nu = -\frac{c}{\lambda^2}\Delta\lambda \qquad \text{... (7.1)}$$

Equation (7.1) expresses the relationship between the frequency spread ($\Delta\nu$) and wavelength spread ($\Delta\lambda$).

For a laser beam, $\Delta\lambda \cong 0.001$ nm while in case of gas discharging in a tube $\Delta\lambda = 0.01$ nm and for a source of white light $\Delta\lambda = 300$ nm. One can minimize this spectral width by using different modern techniques. To become perfectly monochromatic a laser in expected to have $\Delta\nu = 0$.

(b) High Degree of Coherence of Laser

Laser waves are highly coherent. So, if one wants to understand the laser well, one must aquire some knowledge of coherence. **Coherence is the predictable correlated motion of two waves when there is a fixed amplitude and phase relationship between them. If two light waves coming out of two sources are coherent then these sources are also said to be coherent. In other words, coherence is related to definite phase relationship between two waves (or sources) at different points of space and time.** For a source to be coherent, it must be able to emit radiations of single frequency or the frequency spread must be very small. The wavefronts spreadings from the source must also be able to maintain a constant shape. Thus, we can categorize the coherence into two categories: (i) temporal coherence, and (ii) spatial coherence.

(i) Temporal Coherence If one is able to predict the phase and amplitude of a wave at a given point in space at different instants of time or if two waves always maintain a fixed phase and amplitude difference while passing through a fixed point in space, then such waves are called temporally coherent ones and the phenomenon is known as **temporal coherence**. The definition given above is an ideal one. A real source of light emits light in short pulses known as **wavetrains**. As for example, when an excited atom of a substance goes from a higher state to a lower state, it emits a pulse of light during a short interval of time (Δt) ranging from 10^{-10}s to 10^{-8} s. For example, in case of red cadmium line (λ = 643.8 nm), $\Delta t \approx 10^{-9}$ s. There cannot be any phase relationship between the pulses which originate from two different atoms of a substance since atoms of a given substance emit radiations randomly. However, one can predict the phase and amplitude of a

wave at a given point in space at two instants of time if the same wavepulse keeps on passing through the said fixed point. This longest time span (t_c) during which one can make a prediction on phase and amplitude of a wave is called **coherent time.** The length of the wavepulse corresponding to t_c is called **coherence length** (l_c). The coherence length l_c is given by

$$l_c = t_c \times c \qquad \text{...(7.2)}$$

where c is the speed of light in the free space.

(ii) Spatial Coherence **The spatial coherence refers to the phase relationship between two waves present at two space points at the same point of time.** If there is a point source S then, two equidistant points P and Q (which are not along the same line with the source S) will always have a definite phase relationship (Fig. 7.1)

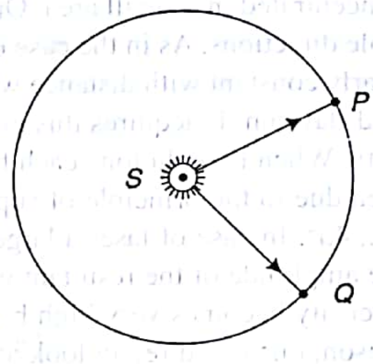


Fig. 7.1 Points P and Q which are equidistant from the source S can maintain definite phase relationship

But if one gradually increases the size of the source S, then after some time the points P and Q will fail to maintain definite phase relationship. The maximum lateral dimension of the source upto which the radiations from it remain coherent determines the spatial coherence. That is, **the spatial coherence depends on the size or dimension of the source concerned.**

Though, theoretically the output of a laser is expected to be perfectly coherent, but in reality, it does not show perfect coherence because all photons are not instigated by the original stimulating photon which initiates the stimulated emission. A detailed discussion on temporal as well as spatial coherence has been made in the first chapter of this book along with interference of light. Interested readers may refer to that chapter. An analytical discussion of coherence has been made in appendix-B

(c) High Directionality of Laser

A beam of conventional light spreads very fast as it passes away from the source, i.e., the rays of the light beam diverge quickly resulting in the rapid increase of the area of the moving wave front. On the other hand, in case of a laser beam, it is observed that the divergence is very less. This property of a laser beam to maintain the parallelity between any two rays of it for a long distance can be termed as its **directionality**. Laser beams are highly directional as compared to those of conventional light. As for example, a laser beam of 10 cm in diameter will not be wider than 5 km in diameter while travelling from the earth to the moon where the distance between the two said celestial bodies is 3,84,000 km. In case of perfectly directional light beam the diameter of the wavefront will remain same. But in reality no light beam is perfectly directional.

The degree of directionality is generally expressed in terms of divergence. As stated above, the divergence gives a measure of spreading of the beam when it is emitted from the source. Using small angle approximation, it can be shown that the laser beam increases in diameter about 1 mm while travelling through a distance of one metre resulting in a beam divergence of 1 milliradian per metre. If one considers two points at distances r_1 and r_2 respectively from the windows of the laser souce and measures the diameters of the laser spot at the aforesaid points and finds the values of the diameter to be d_1 and d_2 respectively, then the angle of divergence in degrees can be expressed as

$$\theta = \frac{d_2 - d_1}{2(r_2 - r_1)} \qquad \text{...(7.3)}$$

The full angle of divergence (θ) in terms of the minimum size of the spot (d_0) is given by

$$\theta = \frac{1.27\lambda}{2d_0} \qquad \text{... (7.4)}$$

where d_0 is the intrinsic size of the beam within the optical cavity of the laser and λ is the wavelength of the radiation emitted from the laser source.

(d) High Intensity or Brightness of Laser

The laser source emits the radiation through its window in the form of a narrow beam having its energy concentrated in a small area. On the other hand, the light from a ordinary light source gets emitted in all possible directions. As in the case of laser, the beam travels in the form of plane wave, the intensity of it remains nearly constant with distance while it travels forward. In fact, a one milliwatt He-Ne laser is brighter than the mid-day sun. It acquires this property because of its other two properties namely – coherence and directionality. When two photons each having amplitude a are in phase with each other and interact with each other, then due to the principle of superposition, the resultant amplitude becomes $2a$. Hence the intensity in $(2a)^2$ i.e., $4a^2$. In case of laser a large number of photons (say n photons) is in phase with each other. As a result, the amplitude of the resultant wave becomes (na) and the corresponding intensity is (n^2a^2) consequently the intensity becomes very high because of coherent addition of amplitudes and negligible divergence. For this reason, one can directly look at a glowing 100 watt electric bulb, but on the other hand one cannot look at a one milliwatt He-Ne laser that has 10^5 times lesser power than that of the electric bulb. A one milliwatt He-Ne laser source is 100 times brighter than the mid-day sun.

7.3 ABSORPTION AND EMISSION OF RADIATIONS BY MATTER

When light energy interacts with matters, both the activities of absorption and emission of photons take place in the concerned matters. By absorbing photons, the atoms of the matters get excited and as the atoms cannot remain in the excited state for a long time, they return to the ground state or for the time being to some metastable states by emitting light.

7.3.1 Absorption of Radiation

The atoms of the active medium have a number of discrete energy states called quantised energy states. They are characterized by their principal quantum number n (where n is an integer). These atoms of the medium usually remain in the ground state with a minimum energy E_1 in absence of any external influence. When they are subjected to some interaction, e.g., irradiation by light photons of frequency ν, they transit to some higher energy state E_2, by absorbing the energy $h\nu$ of the interacting photons. This phenomenon is known as absorption or excitation of the atoms of the active medium. In fact, it is a stimulated or induced absorption because absorption is necessarily a stimulated or induced process (Fig. 7.2). The frequency of such a transition is given by

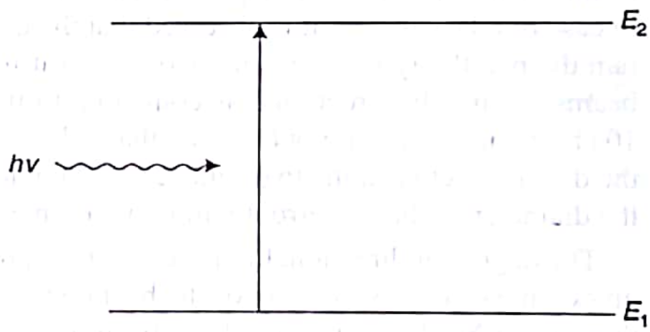


Fig. 7.2 An atom transits from the ground state E_1 to a higher state E_2 by absorbing a photon of energy $h\nu$.

$$\nu = \frac{E_2 - E_1}{h} \quad \text{...(7.5)}$$

7.3.2 Spontaneous Emission of Radiation

Let us now consider an atom which is initially in the excited state E_2 (Fig. 7.3). As an atom cannot stay in an excited state for a long time it returns, of its own, to the initial state E_1 by emitting a single photon of

frequency v after a time interval of about 10^{-8} s or it returns to the ground state in two or more steps by emitting photons of energy v' (where $v' < v$) one after another via the metastable states of intermediate energy E' where $E_1 < E' < E_2$.

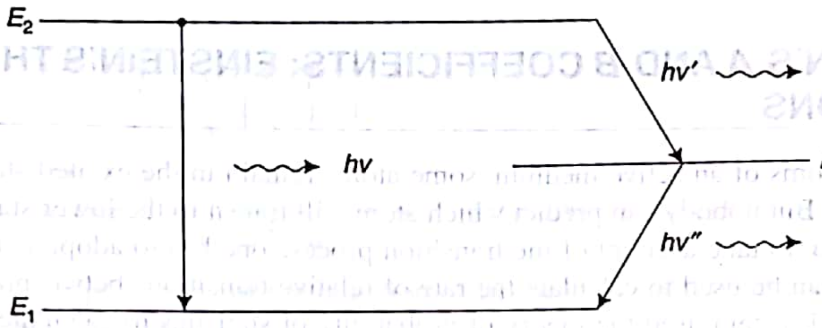


Fig. 7.3 Spontaneous emission. An excited atom transits from the excited state E_2 to the ground state E_1 directly or via a metastable state E'.

This process of returning to the initial state E_1 is called **spontaneous emission or de-excitation.** And the energy of the emitted photon is given by

$$hv = E_2 - E_1 \quad \text{or} \quad hv' = E_2 - E' \qquad \text{... (7.6)}$$

In fact, the active material under observation contains an assembly of atoms and the photons emitted spontaneously by the atoms (due to the process of spontaneous transition) have no directivity. They are emitted randomly in different directions with random or uncorrelated phase. Therefore, the light emitted by the atoms of the sample under observation is non-coherent in nature.

7.3.3 Stimulated Emission of Radiation

The quantum description of the processes of spontaneous absorption and spontaneous emission is identical—just a transition from one level to another level of energy—though as a result of the former the atom gets excited and as result of the later the atom gets de-excited. But when a photon of precise frequency v (or precise energy $hv = E_2 - E_1$) interacts with an atom which is already in the excited state E_2, it fails to excite the atom concerned as it is already excited. Instead, it produces the following equivalent effect, i.e., it de-excites the atom and brings it back to the ground state. So, under the influence of the electromagnetic field of the interacting photon of frequency v, it makes a transition to the lower energy state E_1 and emits and additional photon of the same frequency v (Fig. 7.4). So, now there are two photons of frequency v, the original one and the one which has been emitted through the process of present influenced emission of radiation in contrast to the spontaneous emission. **This kind of transition is known as induced or stimulated emission of radiation.** It is to be noted that in the spontaneous emission, photons get emitted in random direction while in case of stimulated emission, the light photon always gets released from the atom in the same direction in which the stimulating incident photon moves. It is also noteworthy that both the incident and the stimulated photons are coherent and contribute equally to amplify the incident beam. Now if these two photons interact with

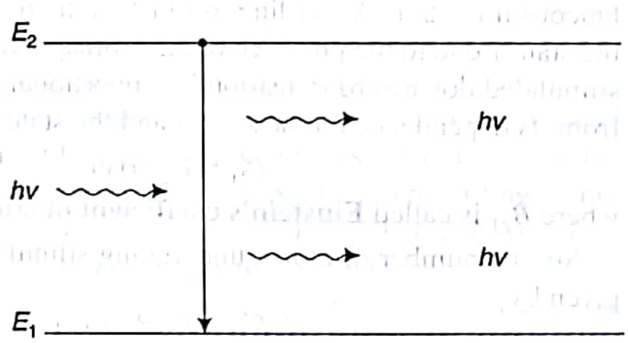


Fig. 7.4 Stimulated emission – a photon of energy hv ($= E_2 - E_1$) interacts with an excited atom and causes it to emit the absorbed energy hv.

two other atoms in the excited state E_2, it will result in stimulated emission of two more photons and so on. Ultimately, the process will form a geometric progression or chain. So, if a lage number of excited atoms get involved in the process, the stimulated emission of photons can generate an intense beam of light having very high coherence and directionality.

7.4 EINSTEIN'S *A* AND *B* COEFFICIENTS: EINSTEIN'S THEORY AND RELATIONS

If in an assembly of atoms of an active medium, some atoms remain in the excited state, then they will de-excite at a certain rate. But nobody can predict which atom will transit to the lower state first. In fact, this is a statistical process. So, to take account of the transition process one has to adopt a statistical method. The theory of probability can be used to calculate the rate of relative transitions between any two energy levels with a good accuracy. Einstein used the theory of probability of statistics to calculate the rate of transition assuming the atomic system to be in equilibrium with an electromagnetic radiation.

7.4.1 Derivation of Einstein's Relations

Let us consider an assembly of atoms which is in thermal equilibrium at a certain temperature T with radiation of frequency ν and energy density $u(\nu)$. The energy density $u(\nu)$ is the amount of energy per unit volume of the active medium which is due to the photons of energy $h\nu$. Let N_1 and N_2 be the number of atoms per unit volume of the active medium respectively in energy levels 1 and 2 at an instant of time. The rate of transition of atoms from state 1 to state 2 (P_{12}) in the process of absorption depends on the said states and it is also proportional to the energy density $u(\nu)$ of the radiation.

$\therefore$ $$P_{12} = B_{12}\, u(\nu),$$

where the proportionality constant B_{12} is called **Einstein's coefficient of absorption** of the radiation.

So, the number of atoms in state 1 that absorb a photon and transits to the state 2 per unit time is given by

$$N_1 P_{12} = N_1 B_{12}\, u(\nu)$$

The probable rate of spontaneous transitions from state 2 to the state 1 depends on both the state 1 and the state 2 and does not depend on the energy density $u(\nu)$. So, the number of atoms in the state 2 which transit to the state 1 by the process of spontaneous transition per unit time is given by

$$N_2 P_{21} = N_2 A_{21}$$

where A_{21} is called **Einstein's coefficient of spontaneous emission** and P_{21} is the rate of downward spontaneous transition. According to Einstein, there is again a downward stimulated transition from the state 2 to the state 1 due to the presence of electromagnetic radiation in the assembly of atoms. The probability of such stimulated downward transition is proportional to the energy density $u(\nu)$ of the stimulating radiation, apart from its dependence on the state 1 and the state 2 and it is given by

$$P'_{21} = B_{21}\, u(\nu)$$

where B_{21} is called **Einstein's coefficient of stimulated emission.**

So, the number of atoms undergoing stimulated transition from the state 2 to the state 1 per unit time is given by

$$N_2 P'_{21} = N_2 B_{21}\, u(\nu)$$

So, at the state of equilibrium, the absorption and emission per unit time must be equal.

Hence, $$N_1 P_{12} = N_2 P_{21} + N_2 P'_{21}$$

or, $$N_2 P_{21} + N_2 P'_{21} = N_1 P_{21}$$

or, $N_2 A_{21} + N_2 B_{21} u(\nu) = N_1 B_{12}\, u(\nu)$

Hence, $$u(\nu) = \frac{N_2 A_{21}}{N_1 B_{12} - N_2 B_{21}}$$

or, $$u(\nu) = \frac{A_{21}}{B_{21}} \cdot \frac{1}{\frac{N_1}{N_2}\left(\frac{B_{12}}{B_{21}}\right) - 1}$$

or, $$u(\nu) = \frac{A_{21}}{B_{21}} \cdot \frac{1}{(B_{12}/B_{21}) e^{h\nu/(kT)} - 1} \quad (\because N_1/N_2 = e^{h\nu}/(kT)) \qquad \text{...(7.7)}$$

Now, this relation of energy density of the radiation of frequency ν must be in accord with the Planck's formula for radiation which is given by

$$u(\nu) = \frac{8\pi h \nu^3}{c^3} \cdot \frac{1}{e^{h\nu/(kT)} - 1} \qquad \text{...(7.8)}$$

Now, having compared Eqs (7.7) and (7.8), we get the following two relations:

$$B_{12} = B_{21} \quad \text{and} \quad \frac{A_{21}}{B_{21}} = \frac{8\pi h \nu^3}{c^3} \qquad \text{...(7.9)}$$

Thus, from the above two relations, we can see that the probability of stimulated absorption is equal to the probability of the stimulated emission. And the second equality implies that the probability of spontaneous emission increases as ν^3, i.e., rapidly with energy difference between the two involved states.

The A's and B's are respectively called Einstein's A and B coefficients. These coefficients cannot be determined with the help of classical electromagnetic theory. But the coefficient B can be determined through quantum mechanical process by making use of Dirac's theory and hence the value of the coefficient A can be calculated. The reciprocal of A_{21} can be taken as a measure of the life-time of the upper state against the spontaneous transition to the lower or ground state.

7.5 WORKING PRINCIPLE OF LASER

The working of laser depends on the phenomenon of stimulated emission of radiation. As was stated earlier, the theory of stimulated emission of radiation was developed by Einstein in 1917. He was considering the equilibrium of matter and electromagnetic radiation in a chamber of black body at a constant temperature where exchange of energy occurs due to absorption and spontaneous emission of radiation by atoms. It was observed by him that the absorption and emission processes together fails to explain the equilibrium situation. There is a difference between the two. In order to explain the cause of the difference between the said absorption and emission processes, he predicted that there must be an additional process of radiation in the blackbody chamber and then termed it as 'stimulated emission'. But his aforesaid prediction regarding stimulated emission attracted little attention of the people engaged in research until 1954. In 1954, one American physicist C H Townes and two Russian physicists N Basov and A M Prokhorov discovered independently and almost simultaneously the phenomenon of construction of maser using ammonia gas. Townes and Schawlow in 1958 successfully showed that the maser principle can also be extended to the field of visible light radiation. And finally in 1960 Maiman first built the device of laser using ruby as the active medium.

If one considers the interaction of the atoms or molecules of an active medium with the radiation them for the interaction to take place, the energy carried by the photon of the interacting radiation ($h\nu$) must be equal to the difference of energy between the two energy states of the concerned atoms or molecules. There are altogether three possibilities for the transitions of the electrons of participating atoms or molecules between two energy levels, namely, (a) induced absorption (or stimulated absorption or simply absorption), (b) stimulated emission, and (c) spontaneous emission.

7.6 POPULATION INVERSION IN LASER

By the term 'population' we mean number of atoms which remain in particular energy state or level, e.g., ground state, first excited state etc. Normally, in the thermal equilibrium state the higher energy levels are much less populated than the lower ones since by Boltzmann law we have $N_2/N_1 = e^{-(E_2 - E_1)/(kT)}$ and $E_2 > E_1$; so, $e^{-(E_2 - E_1)/(kT)} < 1$.

Hence, $N_2 < N_1$. Now, if even $B_{12} = B_{21}$, the stimulated emission is normally unlikely to occur because the higher energy level is less populated. But we can initiate and sustain it; only if we can make a large number of atoms available in the concerned excited energy state by using some means, i.e., if we can make $N_2 > N_1$.

When in a laser active medium, the number of atoms in the higher energy state exceeds the number of atoms in the lower energy state, the active medium is said to have population inversion condition. And the process of obtaining population inversion is called pumping.

7.6.1 Various Methods of Population Inversion

Many different methods of pumping are available. One can adopt either of them for achieving population inversion. The main principle is to selectively transfer a good number of atoms from the ground state to a higher energy state of the atoms so that the number of atoms in the higher state becomes bigger than that of the ground or any other lower energy state. We are selectively discussing some of the pumping methods here.

(a) Optical Pumping In this method of pumping a high energy light source, e.g., Xenon flash lamp, is used. It can supply sufficient photon-flux in a given spectral range to excite atoms. The atoms of the active medium are allowed to rise to the appropriate excited state from the lower state through the process of selective absorption of radiation. These excited atoms then make non-radiative transition through which some energy is transferred to lattice thermal motion. And the atoms reach a metastable state where they remain for a relatively longer time. Thus, one gets population inversion resulting the number of atoms in the excited state more than the member atoms in the ground state. Optical pumping method is used in case of ruby laser.

(b) Excitation by Electrons In this method of pumping, the electrons are released from the atoms of laser active gases by passing high voltage electrical discharge through the gas. The intense electrical field accelerates these electrons to high speeds and they collide with neutral atoms in the gas. A fraction of the gas atoms make thereby a transition to the excited state. This method of pumping is used in some gas lasers.

(c) Inelastic Collision of Atoms When an electric discharge passes through a gas having two types of atoms X and Y, some of the former are raised to excited state X^* and some of the latter to the excited state Y^*. Let one of them (X^*) be in a metastable state – having a mean lifetime longer than normal mean lifetime (approximately 10^{-8} s) of an excited atom of X. If the excitation energies of both types of atoms are nearly equal, then some Y-atoms in the ground state may transit to the excited state through inelastic collision with X^* atoms:

$$X^* + Y \rightarrow Y^* + X$$

and the number of Y^*-atoms continually increases because of the property of metastability of X^* atoms. This pumping is used in case of helium-neon (He-Ne) laser.

(d) Chemical Pumping If an atom or a molecule is produced through some chemical reaction and remains in an excited state at the time of production (e.g., HF), then it can be used for pumping. The HF-molecule is produced in an excited state when hydrogen and fluorine gas chemically combine. The number of so produced excited atoms (or molecules) may considerably be more than the number of atoms (or molecules) in the normal state and hence one can get a population inversion in this process. As an example, we can consider the following chemical reaction:

$$H_2 + F_2 \rightarrow 2HF$$

This chemical reaction generates sufficient energy to pump a CO_2 laser.

(e) Thermal Pumping Sometimes we can obtain a population inversion in an active lasing medium by heating the said medium. As in this method population inversion is achieved by supplying heat, it is called thermal pumping.

7.7 THE BASIC COMPONENTS OF LASER SYSTEM

Structurally a laser device may have the following three main components along with some other less important components: (a) an active medium (in the state of either solid or liquid or gas), (b) A source of pumping (for population inversion), (c) Optical resonator (for causing oscillation). Let us discuss the basic laser system with a block diagram in the following subsection.

7.7.1 The Basic Laser System

Irrespective of the active medium (solid dielectric, semiconductor, liquid or gas) a basic laser system should have at least the three components as stated above. Figure 7.5 shows the block diagram of the basic laser system.

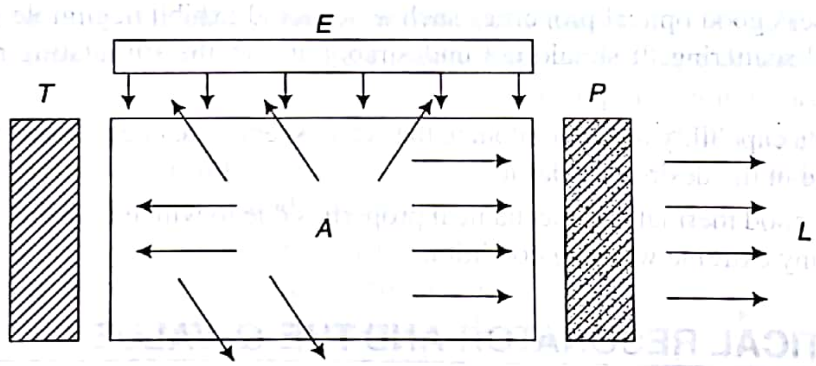


Fig. 7.5 A basic (simple) laser system.

T : Totally reflecting mirror, *P* : Partially reflecting mirror,
E : Exciting source (pumping), *L* : Laser beam (outgoing)
A : The active medium.

The active lasing medium *A* (which is either a solid or a liquid or a gas) is capable of sustaining the stimulated emission. The pumping device or external source of excitation, *E* supplies pumping energy to cause

population inversion in the active medium. The optical resonator consisting of two mirrors *P* and *T* is responsible for the capacity of the laser system to sustain the laser oscillation, a part of the output must be fed back into the laser system. Such a positive feedback is brought about by placing the active medium *A* between the said plane mirrors *P* and *T*, facing each other. The mirror *T* is totally reflecting while the other mirror *P* is partially reflecting. So it allows a part of the generated laser beam *L* to pass through. The arrangement of the two mirrors *T* and *P* is known as the **optical resonator.**

To cause population inversion in the system, the medium *A* is fed with pump energy from *E*. Then this energy is released from the excited atoms by the process of stimulated emission. One photon released by one excited atom stimulates another atom it encounters in the path of it to release a second photon; now these two coherent photons add completely to a beam of double intensity. While the beam moves to and fro through the active medium, its intensity gets rapidly inhanced by more and more stimulated emissions. The reflector mirror *T* gets the beam reversed and allow it another passage (in fact a number of reversions occurs) through the excited medium for further amplification while the beam reaches *P*, a part *L* of the beam escapes as a laser beam. The stimulated photons, which are emitted obliquely with respect to the axis, get lost through the sides of the lasing system so that the final emergent beam is always along the axis of the lasing system.

7.7.2 Characteristics of the Active Medium

The active lasing medium is very much important for a laser system. So we are throwing some more light on its characteristics. This medium consists of an assembly of atoms, molecules or ions (in the form of solid, liquid or gas) which possessed the capability of amplifying light waves. And in normal situation, there always remain a larger number of atoms or molecules in the lower energy levels as compared to the number of such partcles remaining in the higher excited energy levels. While passing through such an assembly of aforesaid particles, an electromagnetic radiation would get attenuated due to absorption of energy of the electromagnetic wave. And in this process, the energy density of the radiation decreases. Thus, to achieve amplification of the electromagnetic radiation, the active lasing medium must be kept in a state of population inversion. In order to obtain population inversion, the active medium must have the following properties:

(a) It should be transparent to both of the stimulating radiation and the laser output,

(b) It should possess good optical properties such as it should exhibit negligible variation in the refractive index and scattering. It should not undesirably absorb the stimulating radiation and the laser output,

(c) It should have a capability to accommodate the active species that help in lasing action without making any change in the desired properties,

(d) It should have good thermal and mechanical properties due to which deformation or fracture will not occur during any extreme working condition.

7.8 THE OPTICAL RESONATOR AND THE *Q*-VALUE

An optical resonator is used in a laser system to sustain the lasing action for a long time through oscillations of the light rays. The optical resonator can have a number of different types of configurations. Out of them, the simplest one has already been discussed in the earlier section (Fig. 7.5). Such a simple device is not used nowadays because with such a device, the required critical precise alignment of the mirrors cannot be obtained. Out of the other alternative configurations (Fig 7.6), we can mention the following four configurations:

(a) the confocal configuration;

(b) the spherical or the concentric configuration:

(c) the hemispherical cavity configuration; and

(d) the long radius cavity configuration.

In the confocal configuration, two identical concave mirrors are placed face to face and the distance between their centres is made equal to the radius of them. In the spherical configuration also, two identical concave mirrors of same radius are placed face to face and the separation of them is made equal to twice of their radius. In the hemispherical configuration, one concave mirror is placed at one end and a plane mirror is placed at its centre of curvature. And in the case of long-radius cavity configuration, two concave mirrors of long but equal radius are placed face to face as shown in Fig. 7.6.

It has been observed that in the confocal configuration the alignment problem is much less. In the hemispherical cavity configuration, though, it is easier to align but the power output of this configuration is comparatively low. The long-radius cavity configuration is most frequently used for commercial lasers.

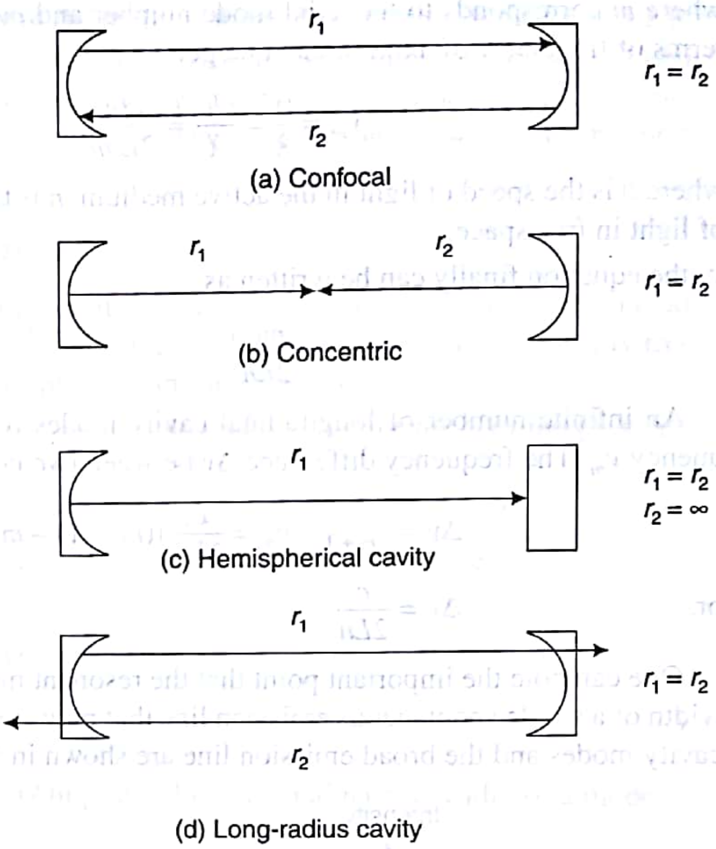


Fig. 7.6 Four configurations of mirrors used in resonators (a) confocal, (b) concentric, (c) hemispherical cavity and (d) long radius cavity.

7.8.1 Generation of Standing Wave in the Resonator

Though for lasing action population inversion is essential but it is not only the decisive factor for the same. A large number of atoms or molecules undergo spontaneous transition and each so generated photon is capable of producing stimulated transitions. These spontaneous phtons have different phase, directions and polarizations. So these photons cannot contribute in the coherent amplification. In order to obtain a coherent emission, the number of photon states must be kept under control. The selectivity of proper photon states and positive feedback are necessary for offsetting the lasers. It can be obtained by enclosing the radiation in a cavity or resonator which is tuned to desired frequency.

Two plane parallel mirrors at the two ends of the active material play the role of resonator. If sufficient population inversion occurs in the active medium, then the light is amplified by the laser action. The light is reflected back and forth by the two mirrors and standing waves are generated. These stationary waves have node at each mirror. When the cavity resonates, the distance L between the mirrors has clearly an integral multiple of half-wavelengths (Fig. 7.7).

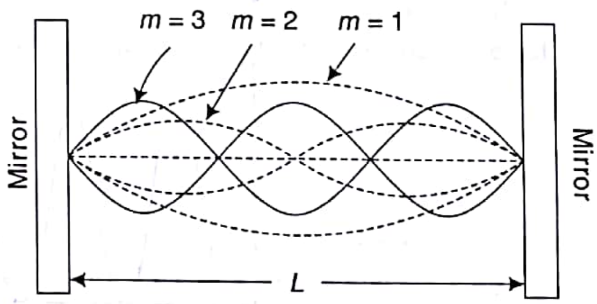


Fig. 7.7 An optical resonator with two parallel mirrors which are L distance apart from each other.

$$\therefore \quad L = m\left(\frac{\lambda}{2}\right),$$

where m corresponds to the axial mode number and $m = 1, 2, 3, ...$ If one expresses the above equation in terms of frequency of axial mode, one gets,

$$\nu_m = \frac{v}{\lambda} = \frac{c/n}{\lambda} = \frac{c/n}{2L/m}$$

where v is the speed of light in the active medium, n is the refrective index of the medium. And c is the speed of light in free space.

$\therefore$ the equation finally can be written as

$$\nu_m = \frac{mc}{2Ln} \qquad ...(7.10)$$

An infinite number of longitudinal cavity modes for oscillations are possible, each with a distinct frequency ν_m. The frequency difference $\Delta\nu$ between two consecutive modes is given by

$$\Delta\nu = \nu_{m+1} - \nu_m = \frac{c}{2Ln}[(m+1) - m]$$

or,

$$\Delta\nu = \frac{c}{2Ln} \qquad ... (7.11)$$

One can note the important point that the resonant modes of the cavity are narrower in frequency than the width of a single spontaneous emission line that may engulf several resonant modes of the cavity. The narrow cavity modes and the broad emission line are shown in Fig. 7.8.

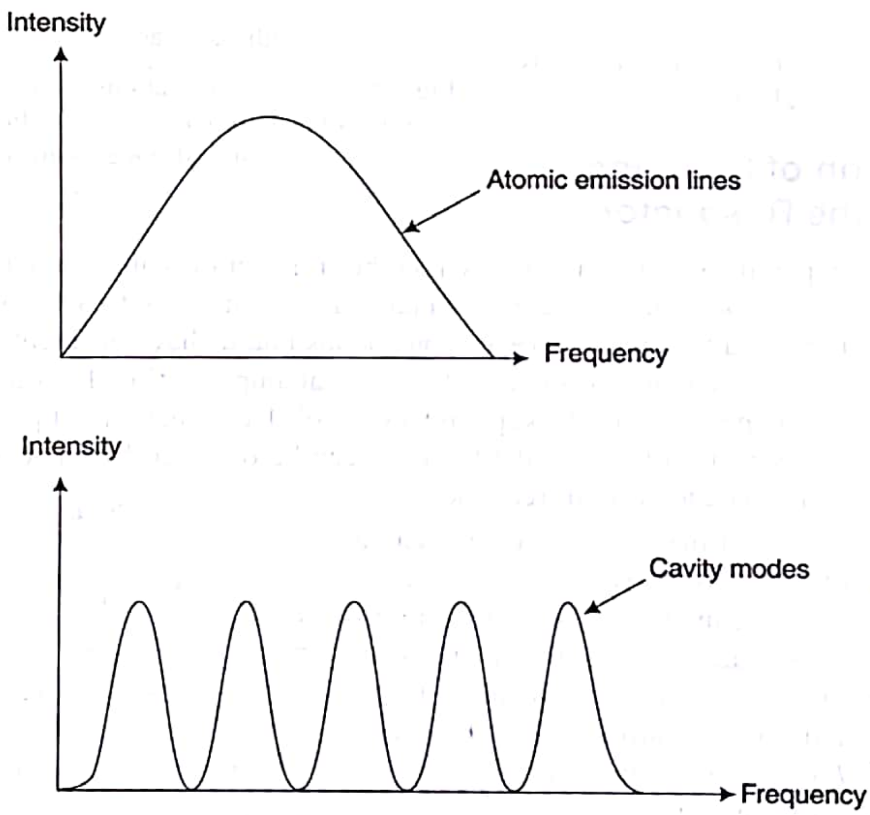


Fig. 7.8 Atomic emission line and cavity modes are shown for an optical resonator.

From the broad range frequencies of an emission line, the cavity mode selects and amplifies only certain narrow bands. It is possible to choose, by changing the separation of the mirrors, only one mode in the band-

width of an emission line. And this explains the property of the extreme monochromaticity of the laser beam. The laser cavity may be either stable or unstable. In the stable resonator, the beam is close to the axis and bounces back and forth between the two mirrors without much energy loss. On the other hand, in an unstable resonator the beam moves away from the axis at each reflection and leaves the resonator within a very short time.

7.8.2 The *Q*-value or the Quality Factor

The resonator does not have sharp line resonant mode. In fact, it has a finite, though small, frequency spread or frequency band. The various losses which are involved with the modes in cavity are linked to this frequency band. The usual losses are principally due to absorption, scattering and diffraction.

A measure of these losses can be expressed in terms of a factor known as quality factor Q or the Q-value of the cavity. Q is defined as

$$Q = 2\pi \times \frac{\text{maximum energy stored per cycle in the concerned mode}}{\text{energy dissipated per cycle in the concerned mode}}$$

in the mode under consideration.

Another alternative expression for the quality factor Q is given by

$$Q = \frac{W}{\Delta W} \qquad \text{.... (7.12)}$$

where ΔW is the line width, so that a high quality factor Q implies a low loss and narrow width of a mode.

7.9 THE THRESHOLD CONDITION FOR SUSTAINING OF LASER ACTION

When population inversion is produced in the cavity, light amplification takes place in a laser system. A laser consists of an active medium kept in between two mirrors which form the optical resonator of the laser system. The light radiation which bounces back and forth within the resonator is amplified by the active medium. The light radiation also suffers some losses due to various reasons. These losses are to be sufficiently compensated by induced transitions in the active medium for the laser action to sustain. Those losses which draw more attention are due to absorption by the active medium, transmission, scattering and diffraction by the mirrors, scattering by optical inhomogeneities of the medium, etc., for the compensation, as has been reffered to above, a minimum strength of population inversion is required. And this **minimum population inversion is called threshold population inversion**. The density of atoms in the higher energy state (i.e., excited state), when population inversion is achieved, is called population inversion density. Different materials have different threshold values of population inversion density as it is related to the intrinsic properties of the material concerned.

7.10 TYPICAL LASERS

Many types of lasers are nowadays being used. They can be classified broadly into four categories: (a) solid dielectric laser, (b) gas laser, (c) semi-conductor laser and (d) tunable laser. They emit red, blue, blue-green or invisible radiation ranging from microwave to ultraviolet. Some of them produce continuous waves (cw), and others produce flash or pulse laser. The pulse rate is very short. It may be in the order of milliseconds, nanoseconds or picoseconds. We shall describe only two lasers here: (a) pulsed ruby laser, and (b) continuous helium-neon gas laser.

7.10.1 Pulsed Ruby Laser

The first successful laser built by Maiman (1960) was ruby laser. It is a solid state laser and was built by using a ruby crystal (i.e., a Al_2O_3 crystal dopped with Cr^{3+} ions at a concentration of nearly 0.05% by mass).

Construction of the Lasing System Doped ruby crystals as stated above are used as active medium. These crystals are grown in special furnaces and then given a shape of a cylinder about 1 cm in diameter and 5 cm in length (Fig. 7.9). The end faces *A* and *B* are polished in such a manner that they become optically flat and parallel.

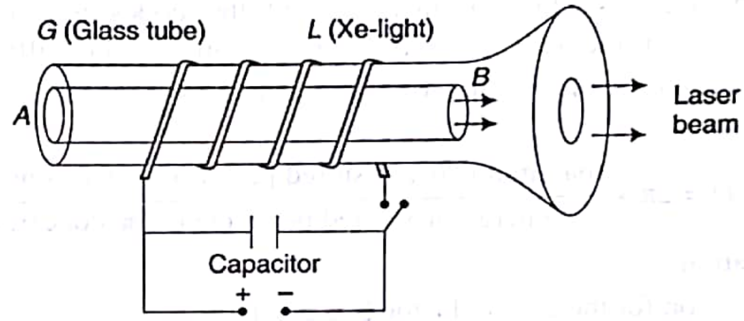


Fig. 7.9 Ruby laser system. *AB* is cylindrical ruby rod, *A*–totally reflecting face, *B*–partially reflecting face.

The two ends are silvered in such a way that face *A* reflects light totally and *B* reflects partially. As the face *A* reflects light totally, it forces the radiation to proceed backward to *B*, which being semisilvered, allows a small fraction (1%) of the intense radiation to emerge out of the laser cavity as laser beam. And the major part of radiation (99%) is reflected back into the crystal. The cylindrical surface is also coated with silver to make it reflecting. A xenon flash lamp *L* (helical in shape) surrounds the ruby rod. The output of the lamp is rich in yellow-green light and it is flashed for optical pumping. A cylindrical glass tube *G* through which liquid nitrogen circulates also surrounds the rod to keep the rod cool.

Working of Ruby Laser When the xenon lamp *L* is switched on, a flash of light from it excites a large number of chromium ions from the ground state E_1 to the wide-band and excited state E_3 by absorption of the light of wavelength $\lambda = 550$ nm (Fig. 7.10); as can be seen in the diagram, the chromium ions have an intermediate metastable state E_2. Soon the excited chromium ions decay to the metastable state E_2 through radiation less de-excitation process. And stay therefore relatively longer time (several milliseconds) to transit

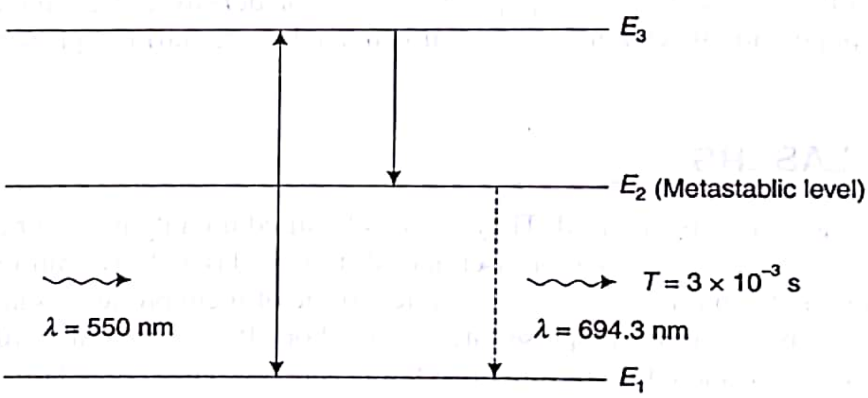


Fig. 7.10 The energy level diagram of ruby laser

the ground state by spontaneous emission of wavelength $\lambda = 694.3$ nm. The lamp L through its flashing, thus increases the population in the metastable state E_2 via the wide-band state of energy E_3. The state E_2 being metastable one, the chromium ions cannot stay in this state for long time. They emit photons spontaneously when one or two photons are emitted spontaneously by them, the photon(s) induce stimulated emission and coherent laser beam in obtained through the partially silvered face B.

7.10.2 Continuous Helium–Neon Laser

Ali Javan and his associates developed the helium–neon (He–Ne) Laser in 1961. It was the first gas laser. He – Ne laser is one of the most popular continuous wave lasers.

Construction of the Lasing System It consists a discharge tube about 1m long, 4–6 mm in diameter and filled with a mixture of pure He – Ne gas in the ratio of 5:1 under the total pressure of 1 torr (Fig. 7.11). The ratio of 5:1 is rather critical. The gain of this laser system is proportional to the tube length. The intensity varies with diameter.

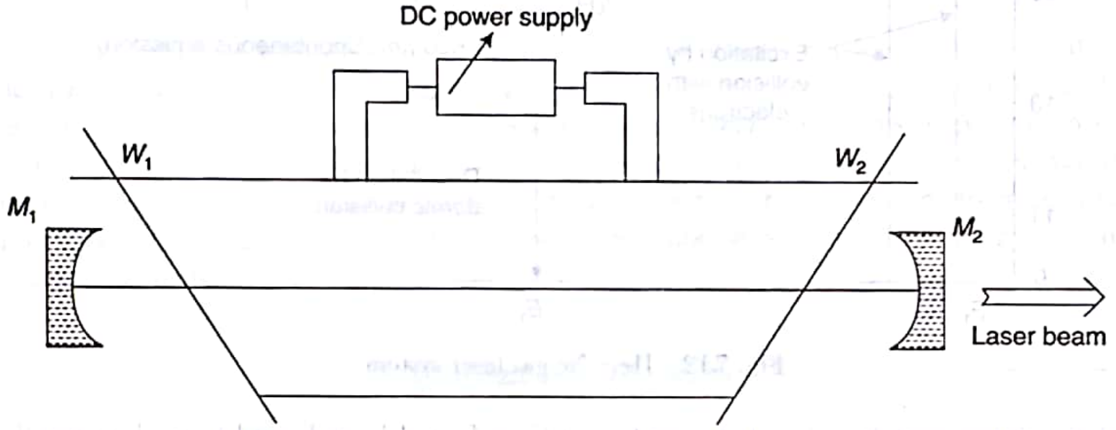


Fig. 7.11 He–Ne gas laser system.

The relative intensity of the laser depends to a great extent on it. The other parts of the He – Ne laser system are one high-voltage dc power supply, two concave spherical mirrors M_1 and M_2, mirror M_1 having 99.5% reflectivity and mirror M_2 being partially reflecting. Each of the mirrors has a radius of curvature equal to the length of the cavity and mounted on a rotatable support to bring them into alignment. Two optically flat windows W_1 and W_2, called Brewster's windows, are fitted by cementing to the ends of the tube at Brewster's angle θ (= $\tan^{-1} \mu$), where μ is the refractive index of the material of the window, to provide a plane polarized beam of light. The window W_1 is fully reflecting while W_2 is partially reflecting. The laser operates continuously at low power (in the range between 5 and 10 W). The output lies between 1 mW and 50 mW and it gives laser of wavelengths 632.8 and 115.2 nm.

Working Principle of He – Ne Laser In order to understand the operation of He – Ne laser, let us take help of the energy-level diagram as shown in Fig. 7.12. Due to discharge in the mixture of the two gases, some atoms of helium are raised by electron collision to the metastable state F_2 and F_3 and the number of so excited helium atoms gradually builds up as no radiative transition to the lower states is possible. Now, some of the excited states of the neon gas are lower than but very close to the aforesaid metastable states of the helium gas. For this reason, when atoms of helium gas in states F_2 and F_3 collide with those of neon gas in the ground state E_1, exchange of energy takes place among them. As a result of this type of collisions between atoms of the two gases, some neon atoms get excited and raised to levels E_4 and E_6 and returning of

the helium atoms to the ground state through de-excitation occurs. The process of energy transfer has a high probability as the states F_2 and F_3 are unstable. The discharge in the cavity (i.e., the gas mixture tube) thus continuously increases the population of neon atoms in excited states E_4 and E_6.

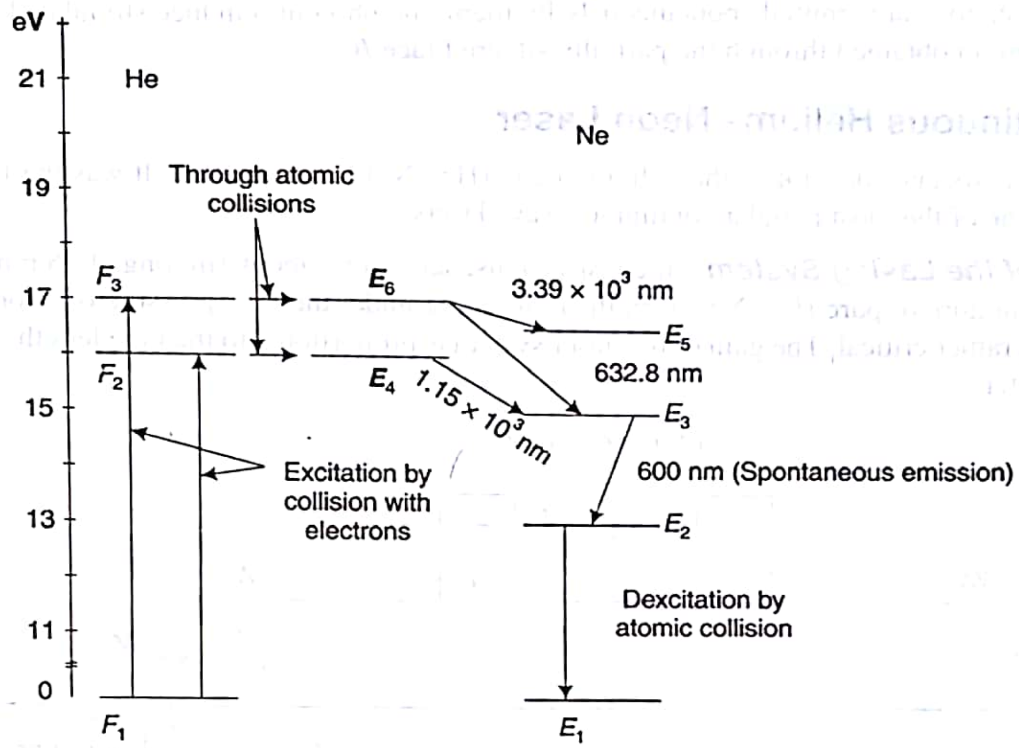


Fig. 7.12 He – Ne gas laser system.

So, population inversion takes place between E_4 and E_3 or E_6 and E_5 or E_6 and E_3 as shown in the diagram by arrowheads. The various transitions between different pairs of layers lead to the emission of radiations of wavelength 3.39×10^3 nm, 1.15×10^3 (both lie in infrared region) and 632.8 nm. The last is the well-known red-light laser. The excited neon atoms then fall from the level E_3 to E_2 by spontaneous transition resulting in the emission of photons of wave-length $\lambda = 600$ nm. The He – Ne laser is more directional and more monochromatic than any solid laser as it is a gas laser. The solid lasers suffer from many imperfections in solids and the heating effect of the flash lamp used while the He – Ne laser does not suffer from such imperfection.

7.11 APPLICATIONS OF LASER

A laser beam possesses a good number of mentionable properties which are usually not found in the light beams obtained from other sources. It is these properties of laser that impart special importance to the laser and because of these specialities of lasers, they have many applications in daily life as well as in science and industries. Let us discuss the applications of laser in the following four categories:

(a) In technical and industrial fields, the laser beam is used for cutting fabrics for clothing in one hand and sheets of steel on the other hand. Laser can be used to make minute holes through drilling in paper clips, single hair strand and hard materials including teeth and diamond. And through such diamond holes extremely fine wires in cables are drawn. Laser welding method can be used to join metallic rods by melting them. Laser heat-treatment is able to harden surfaces of engine crank-shafts

and cylinder walls. Laser beams can be used to vaporize unwanted materials during the time of manufacturing of electronic circuits on the chips of semiconductor.

(b) In the field of medicine, the laser beams are used for 'spot welding' of detached retina, in grafting cornea, in drilling holes in the bones, for destroying specific cancerous areas within tissues of the body in bleeding-free surgery like vocal chord operation, ulcers of stomach, stones in the kidneys, in cutting and sealing small blood vessels during brain operation and also while microsurgery on cells and chromosomes are performed.

(c) In science and for research in science, the laser has become a very useful tool. In Raman spectroscopy, He – Ne lasers are fast replacing mercury arcs with the help of laser multiphoton processes which are being studied in determining the symmetry of wave functions in spectroscopy. Laser beams can be used to confirm the phenomenon of temporal coherence. Laser is also being used in the field of ionization and dissociation studies of gases, in kinetic studies of fast chemical reactions, in spectrochemical analysis of solids in conjunction with mass and emission spectrometer and in the investigation of energy transfer by exciting a specific vibration level and observing its decay in polyatomic molecules.

(d) Other uses of lasers include – 'inertial confinement' of plasma and determination of its temperature; destruction of enemy missiles, to aim at the enemy at night; laser rifles, laser pistols and laser bombs are also being made; detection of earthquakes and nuclear explosions, vaporization of solid fuels of rockets and study of the surfaces of distant planets and satellites, accurate measurement of very long distances, the distance between the earth and moon has been measured to an accuracy of 15 cm. Laser can also be used to make three dimensional motion pictures. Now laser is essential in communication technology.

Worked-out Examples

Example 7.1 Find the difference of energy between two energy levels of neon atom if the transition between these levels gives a photon of wavelength 632.8 nm. Also calculate the number of photons emitted per second to give a power output of 2 milliwatt.

Sol. The energy difference between the said energy levels of neon atom,

$$\Delta E = E_2 - E_1 = h\nu = \frac{hc}{\lambda}$$

or,
$$\Delta E = \frac{6.63 \times 10^{-34} \times 3 \times 10^8}{632.8 \times 10^{-9}} = 3.143 \times 10^{-19} 9\,\text{J}$$

or,
$$\Delta E = \frac{3.143 \times 10^{-19}}{1.6 \times 10^{-19}}\,\text{eV} \quad [\because 1\text{ eV} = 1.6 \times 10^{-19}\text{J}]$$

or,
$$\Delta E = 1.96\text{ eV}$$

If n be the number of photons emitted per second to give a power output of 2 mW, then

$$P = n \times \Delta E$$

or,
$$n = \frac{P}{\Delta E} = \frac{2 \times 10^{-3}\text{ watt}}{3.143 \times 10^{-19}\text{ J}} = 6.363 \times 10^{15}$$

So, the number of photons emitted per second is given by 6.363×10^{15}.

Example 7.2 If the wavelength of radiation of ruby laser is 694.3 nm, what is the photon emitted?

Sol. Given the wavelength of radiation is $\lambda = 694.3$ nm.

The energy of the radiated photon is

$$E = h\nu = \frac{hc}{\lambda} = \frac{6.625 \times 10^{-34} \times 3 \times 10^{8}}{694.3 \times 10^{-9}} \text{ J}$$

or, $$E = 1.787 \text{ eV.}$$

Example 7.3 In a He – Ne laser transition from E_3 to E_2 level gives a laser emission of wavelength 632.8 nm. If the energy of the E_2 level is 15.2×10^{-19} J, how much pumping energy is required, if there is no energy loss in the He – Ne laser ? [WBUT – 2008]

Sol. Wavelength of He – Ne laser is $\lambda = 632.8 \text{ nm} = 6328 \times 10^{-10}$ m.

The energy of the laser is $E' = h\nu = \dfrac{hc}{\lambda}$

or, $$E' = \frac{6.625 \times 10^{-34} \times 3 \times 10^{8}}{6328 \times 10^{-10}} \text{ J} = 3.1 \times 10^{-19} \text{ J}$$

or, $$E' = 1.935 \text{ eV}$$

The energy of the level E_2 is $E'' = 15.2 \times 10^{-19}$ J.

i.e., $$E'' = 9.488 \text{ eV}$$

Now if E be the energy of the pumping source, then $E = E' + E''$.

Hence $$E = 1.935 \text{ eV} + 9.488 \text{ eV.}$$

or, $$E = 11.423 \text{ eV.}$$

Example 7.4 In a lasing process, the ratio of population of two energy states E_1 and E_2 (E_2 being a meta-stable state) is $1:1.009 \times 10^{25}$. Calculate the wavelength of the laser beam at a temperature of 320 K.

Sol. Let N_1 and N_2 be the number of atoms per unit volume of the states E_1 and E_2 respectively, then

$$\frac{N_1}{N_2} = \frac{1}{1.009 \times 10^{25}} \quad \text{or} \quad \frac{N_2}{N_1} = 1.009 \times 10^{+25}$$

And temperature $T = 320$ K.

Now, using the Boltzmann relation, we get

$$\frac{N_2}{N_1} = e^{-(E_2 - E_1)/(kT)} = e^{-\Delta E/(kT)}$$

or, $$\ln\left(\frac{N_2}{N_1}\right) = -\frac{\Delta E}{kT}$$

Now, considering magnitude only, we get

$$\Delta E = kT \ln\left(\frac{N_2}{N_1}\right)$$

or, $$\Delta E = 1.38 \times 10^{-23} \times 320 \times \ln(1.009 \times 10^{25})$$

or, $$\Delta E = 1.38 \times 10^{-23} \times 320 \times 57.57$$

or, $$\Delta E = 2.54 \times 10^{-19} \text{ J}$$

Now, if λ be the wavelength of the laser, then

$$\Delta E = \frac{hc}{\lambda}$$

or, $$\lambda = \frac{hc}{\Delta E} = \frac{(6.63 \times 10^{-34} \text{ J} - \lambda) \cdot (3 \times 10^8 \text{ ms}^{-1})}{2.54 \times 10^{-19} \text{ J}}$$

or, $$\lambda = 7.83 \times 10^{-7} \text{ m}$$

or, $$\lambda = 783 \text{ nm}$$

Example 7.5 Two plane mirrors forming a resonant cavity in a He – Ne laser are at a distance of 0.5 m. Find the frequency separation between two consecutive modes in longitudinal cavity.

Sol. The distance between the two plane mirrors $L = 0.5$ m. So, the difference between the frequencies of two consecutive modes is given by

$$\Delta \nu = \frac{c}{2L\mu}$$

where velocity of light $c = 3 \times 10^8$ m/s

and refractive index of the medium $\mu = 1$

$$\therefore \quad \Delta \nu = \frac{3 \times 10^8}{2 \times 0.5 \times 1} = 3 \times 10^8 \text{ Hz}$$

Example 7.6 A relative population (Boltzmann ratio) of $1/e$ is representative of the ratio of populations in two energy states at room temperature ($T = 27°\text{C}$). Determine the wave-length of the radiation emitted at the temperature.

Sol. With usual meanings of the symbols, we have

$$\frac{N_2}{N_1} = e^{-(E_2 - E_1)/(kT)}$$

But, here, $$\frac{N_2}{N_1} = \frac{1}{e}$$

Hence $$\frac{1}{e} = \frac{1}{e^{(E_2 - E_1)/(kT)}}$$

or, $$\frac{E_2 - E_1}{kT} = 1$$

or, $$E_2 - E_1 = kT \Rightarrow kT = h\nu = \frac{hc}{\lambda}$$

or, $$\lambda = \frac{Ch}{kT} = \frac{(3 \times 10^8)(6.62 \times 10^{-34})}{(1.38 \times 10^{-23})(273 + 27)}$$

or, $$\lambda = 4.8 \times 10^{-5} \text{ m}$$

Review Exercise

Part 1 : Multiple Choice Questions

1. In He – Ne Laser, neon atoms obtain energy (WBUT 2008)
   (a) on collision with helium atoms (b) from chemical reactions
   (c) from electrical pumping (d) optical pumping
2. In a ruby laser, population inversion is achieved (WBUT 2007)
   (a) through optical pumping (b) through inelastic collisions
   (c) through chemical reactions (d) by applying strong electric field
3. Population inversion in preparing laser beam can be achieved (WBUT 2006)
   (a) when one of the excited states is less populated than the ground state
   (b) when one of the excited states is more populated than the ground state
   (c) when the population of one excited state and the ground state are equal
   (d) on the basis of none of the above conditions
4. Emission of photons due to the transition of an electron from a higher to a lower energy state caused by external energy is known as
   (a) stimulated absorption (b) amplified emission
   (c) stimulated emission (d) spontaneous emission
5. The population of electron in different energy states of a system in the thermal equilibrium is governed by
   (a) Bragg's law (b) Einstein relations
   (c) Boltzmann distribution law (d) Wien's displacement law
6. The colour of the laser output in case of ruby laser is
   (a) violet (b) blue (c) red (d) green
7. In case of laser, variation of divergence of a laser beam with distance gives us an idea about its
   (a) brightness (b) coherence (c) monochromaticity (d) directionality
8. The ratio of He and Ne in a helium-neon laser is of the order of
   (a) 1:15 (b) 1:1 (c) 1:10 (d) 5:1
9. In the active medium of ruby laser the percentage of chromium ions in Al_2O_3 is
   (a) 0.5 (b) 0.05 (c) 0.005 (d) 0.0005
10. The wavelength of He – Ne laser is
    (a) 632.8 nm (b) 600 nm (c) 532.8 nm (d) 500 nm
11. The wavelength of ruby laser is
    (a) 694.3 nm (b) 632.8 nm (c) 600 nm (d) 633.3 nm
12. Coherence of light can be measured from
    (a) wavelength of the beam (b) variation of spot size in distance
    (c) brightness of the beam (d) none of these
13. A three-level laser system will be
    (a) always pulsed (b) either *CW* or pulse (c) always *CW* (d) none of these

14. For a laser action to take place, the medium used must have at least
(a) 4 energy levels (b) 2 energy levels (c) 3 energy levels (d) none of these
15. In a gas laser, generally optical pumping is not applied because
(a) absorption band of the medium is narrow
(b) absorption band of the medium is broad
(c) the process is not able to excite atoms in such a higher energy state of gaseous atom
(d) none of these
16. The metastable state has a mean life-time of more than
(a) 10^{-3} s (b) 10^{-5} s (c) 10^{-4} s (d) 10^{-2} s
17. The ratio of Einstein's A and B coefficient, i.e., A_{21}/B_{21} (say) is proportional to
(a) ν (b) ν_2 (c) ν_3 (d) ν_4

Answers

1. (a)	2. (a)	3. (b)	4. (c)	5. (c)	6. (c)	7. (d)	8. (d)
9. (b)	10. (a)	11. (a)	12. (c)	13. (a)	14. (c)	15. (a)	16 (a)
17 (c)							

Short Questions with Answers

1. **What are the differences between ordinary light and laser light?**
The ordinary light may be either polychromatic or monochromatic depending on the nature of sources which produce it while the laser light is perfectly monochromatic irrespective of the source which produces it. The conventional sources produce ordinary light which is basically incoherent because the producing sources emit light of random wavelengths with no common phase relationship. On the other hand, laser light is highly coherent having constant phase relationship among various rays produced by the laser source. Ordinary light sources emit light in all directions whereas laser sources emit light rays in a unique direction making it highly directional. The brightness of a laser beam is much more than that of the ordinary light.
2. **What is working principle of laser?**
Refer to Article 7.5.
3. **What is stimulated emission?**
If an atom is excited, then at least one electron of the atomic shell(s) occupies higher energy level(s). In such a situation an incident photon having energy equal to the difference of the energies of the occupied higher orbit and any other lower orbit (where the number of electrons is less than the required number of electrons for a complete shell), may cause one electron in a higher shell to jump to the said lower shell and as a result of this act a photon is emitted. This kind of emission of photons is called stimulated emission.
4. **What are population inversion and pumping?**
Usually, due to thermal equilibrium, the higher energy levels of atoms are much less populated than the lower energy levels (including the ground level) since by Boltzmann law $N_2/N_1 = e^{-(E_2 - E_1)/(kT)}$ i.e., $N_2 < N_1$ for $E_2 > E_1$. So, even when $B_{mn} = B_{nm}$; the stimulated emission is normally unlikely to occur. It can be initiated and continued; if only by some means a large number of atoms are made to attain in appropriate excited energy levels. The situation in which the number of atoms in the higher

energy level exceeds that in the lower energy level is known as population inversion and the process of obtaining population inversion is known as pumping. Pumping may be done in many ways. Optical pumping, chemical pumping etc. are a few of the various ways of pumping to cause population inversion.

5. **What is optical resonator? Discuss it in brief.**

   For sustaining the laser oscillation, a part of the output must be fed back into the laser producing system (i.e., the laser cavity consisting of the active medium). Such a positive feedback is caused to occur by placing the active lasing medium in between two plane parallel mirrors which face each other. One of them is placed at one end of the active medium and the other one is placed at the other end of the medium. One of the mirrors is partial reflector and the other one is total reflector. The partial reflector allows a part of the generated laser beam to go out of the system. This arrangement is known as the optical resonator. This device is capable of causing a resonance between the stimulating photons and the excited atoms capable of emitting photons of same frequency. Hence the name optical resonator.

6. **What are the characteristic properties of a laser beam?**

   Refer to Article 7.2.

7. **Describe in brief, the working principle of laser?**

   Refer to Article 7.5.

8. **Discuss briefly the application of laser in the medical field?**

   Refer to Article 7.11(b).

9. **Why is helium gas used in the helium-neon laser system?**

   In He – Ne laser the masses of the two types of atoms are not comparable. The atoms of neon are much heavier than those of helium. So, it is not possible to excite the neon atoms easily efficiently with the help of collision with electrons generated from electric discharge in the mixture of neon and helium atoms. At the same time, the possibility of the transfer of energy of an electron which is accelerated through the collision with helium atoms is maximum because they are lighter atoms. There are some metastable states in helium atoms which are almost identical to the two energy states of the neon atoms. Therefore, at first the energy of accelerated electrons is transferred in helium atoms through inelastic collisions, then that energy is transferred to the neon atom through resonance transfer of energy. So, the excitation is transferred to the neon atoms via the helium atoms. The He atoms act as carriers of excitation energy.

10. **What is resonance transfer of energy?**

    If there be two atoms of different kinds with identical or almost identical energy levels, then transfer of energy from one atom to the other is possible when they take part in the event of collisions. This type of energy transfer is called resonance transfer of energy.

Part 2: Descriptive Questions

1. Explain clearly the terms 'absorption' 'spontaneous emission' and 'stimulated emission'
2. (a) Define Einstein's A and B coefficients and deduce their mutual relation.

   (b) Show that the ratio of spontaneous to stimulated emission is proportional to the cube of the frequency.

   (c) Explain the construction and action of the optical resonator in a ruby laser. [WBUT – 2008]

3. Explain the working principle of He – Ne laser with energy level diagram. [WBUT – 2007]
4. What is the full form of laser? Describe different properties of laser. State the difference between laser and ordinary light.
5. What are the different transition process involved with lasing action? Describe with energy level diagram, the phenomena of spontaneous emission, stimulated emission and stimulated absorption in a two-level system.
6. What is population inversion in case of laser? Explain it. Obtain a raltion between Einstein's A and B coefficients. What are their physical significance?
7. What do you understand by a metastable state? What is rule of a metastable state in order to obtain population inversion in an active laser medium?
8. Draw a net sketch diagram of ruby laser. Discuss the operation of a ruby laser with the help of energy diagram.

Part 3: Numerical Problems

1. Calculate the ratio of stimulated and spontaneous emission rates for the wavelength $\lambda = 5893$ nm at 200°C.
2. The energy gap between two levels corresponds to wavelength $\lambda = 500$ nm. Find the ratio of populations of the two states in thermal equilibrium at 27°C.
3. Find the ratio of stimulated to spontaneous emission rate at a temperature of 240°C for sodium D lines.
4. If the wavelength of the radiation of the He – Ne laser is 632.8 nm, calculate the energy of the radiation. **[1.9625 eV]**
5. A normal optical source emits a radiation of wavelength 600 nm at a temperature of 900 K. Show that the stimulated emission in the system is negligible compared to the spontaneous emission.

CHAPTER

8

Holography

8.1 INTRODUCTION

In holography, *holo*–means whole and *graphos*–means writing. Therefore, holography means a whole writing (a Greek word). In 1948, Dennis Gabor invented the principle of holography and received Nobel Prize in the year of 1971 for this work. Gabor introduced a two step lensless imaging process known as wavefront reconstruction technique or holography, in which an interference between the object wave and the background wave (known as reference wave) is formed and recorded on a photographic plate.

The recorded photograph is called a hologram (whole record). The hologram contains information not only about the amplitude (hence intensity) but also about the phase of the object wave. To view the image at later time he used another wave known as reconstruction wave (which in most cases is identical to the reference wave) for illuminating the hologram. Thus, ***holography is a technique of recording the amplitude and phase of the light waves reflected from an object.***

The process of producing a holographic reconstruction can be explained purely in terms of interference and defraction. The principle of holography contains two steps:

(i) ***Recording of hologram:*** The recording of hologram is based on interference of coherent light waves.

(ii) ***Reconstruction of hologram:*** The reconstruction of hologram is based on diffraction of light waves.

8.2 PRINCIPLE OF RECORDING OF A HOLOGRAM

The principle of recording of a hologram includes superposition of scatter wave from the object and another coherent wave known as reference wave of same wavelength as scattered wave (Fig. 8.1). The light from a laser source is split into two components. One of the component wave is directed towards the object is known as ***object wave***. The other one is directed towards the recording plate (photographic plate) is known as ***reference wave***. Upon interference of these two waves in the photographic plate, interference fringes are produced and recorded on the photographic plate. It contains the information about phase and amplitude of the waves. The photographic record of interference pattern generated by an object is called a ***hologram***.

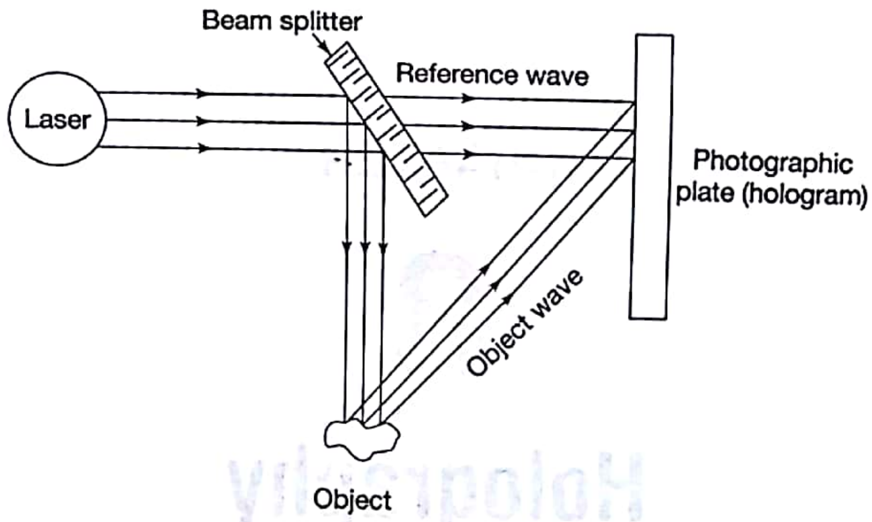


Fig. 8.1 Recording of a Hologram

8.3 VIEWING THE HOLOGRAM (RECONSTRUCTION OF A HOLOGRAM)

The reconstruction wave of same wavelength as reference wave illuminating the developed hologram. The reconstruction wave interacts with the interference pattern on the hologram and gets diffracted to produce two images of the original object (Fig. 8.2). However, the two images differ in appearance to the observer. The virtual image is produced at the same position as the object and has the same characteristics of the object like parallax as the original three dimensional object. One can observe the different perspectives of the object on moving his eye. This image is known as ***orthoscopic image***. The real image can be photographed without the aid of lenses just by placing a light sensitive medium at the position where the real image is formed. The real image is also formed at the same distance from the hologram, but in front of it. The real image will appear inverted in-depth. The real image is known as ***pseudoscopic image***.

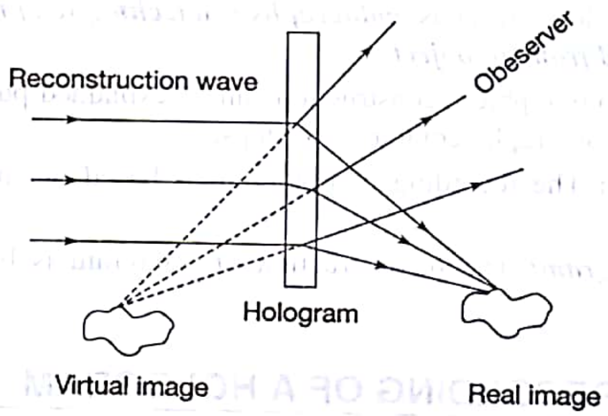


Fig. 8.2 Viewing the image

8.4 CHARACTERISTIC OF A HOLOGRAM

The unique characteristic of holograms are given below:

(i) One real and one virtual images can be reconstructed from a hologram.

(ii) The hologram of a diffuse object (means the wave from each scattering point of the object reaches all parts of the hologram plate) can be reconstructed by a small portion of the hologram. If a hologram breaks into several pieces, then each piece can reproduce the entire image. However, as the hologram site reduces, a loss of image perspective, resolution and brightness result in the constructed image.

For non-diffusely reflecting objects, one makes use of an additional diffusing screen through which the object is illuminated.

(iii) A cylindrical hologram provides a 360 degree view of the object.

(iv) To get a good hologram, the length of the illuminating object beam should be equal to the length of the reflection beam.

(v) The holography system must be absolutely still during exposure. The exposure time is of the order of 5 seconds and depends on the colour of the object.

8.4.1 Hologram Aberrations

Hologram aberrations caused by a change in the wavelength from construction to reconstruction and also by mismatch in the reference and reconstruction waves.

8.4.2 Requirement of an Ideal Display Hologram

An ideal display hologram should have the following requirements:

(i) ***Coherence*:** Both temporal and spatial coherence are important for display hologram.

(ii) ***Resolution*:** The resolution in the reconstructed image should be good.

(iii) ***Aberrations*:** The image must be aberration free.

(iv) ***Field-of-view*:** The field of view should be wide to allow a large parallax.

(v) ***Stability*:** The film, the object and any mirrors used in producing the reference beam must be motionless with respect to one another during exposure.

(vi) ***Noise*:** The background should be noise free. Background noise reduces the contrast of the image.

8.5 THEORY OF HOLOGRAPHY

To explain the theory of holography let us consider the object wave, which is due to the superposition of waves from point scatterers on the object, as

$$O(x, y) = A_1(x, y) \cos(\phi - \omega t) \quad \text{...(8.1)}$$

The object were represented by Eq. (8.1) lies in the plane of photographic plate at $z = 0$ and ω is the frequency of the wave. Holography records the object wave, particularly the phase $\phi(x, y)$ associated with it. Now we consider a plane reference wave propagating in the x-z plane inclined at an angle α with the z-direction. The field associated with the reference wave can be written as

$$\begin{aligned} R(x, y, z) &= A_2 \cos(\vec{k} \cdot \vec{r} - \omega t) \\ &= A_2 \cos(kx \sin\alpha + kz \cos\alpha - \omega t) \end{aligned} \quad \text{...(8.2)}$$

At the photographic plate, i.e., at $z = 0$, the field becomes

$$R(x, y) = A_2 \cos(kx \sin\alpha - \omega t) \quad \text{...(8.3)}$$

We know that the propagation constant $k = \dfrac{2\pi}{\lambda}$, so $kx \sin\alpha = \dfrac{2\pi}{\lambda} x \sin\alpha$. Here, $\dfrac{\sin\alpha}{\lambda}$ is defined as the spatial frequency (say, ξ)

Now the reference field becomes.

$$R(x, y) = A_2 \cos(2\pi\xi x - \omega t) \quad ...(8.4)$$

The total field at the photographic plate (at $z = 0$) is given by

$$T = O + R$$

or,
$$T(x, y, t) = A_1(x, y) \cos(\phi - \omega t) + A_2 \cos(2\pi\xi x - \omega t) \quad ...(8.5)$$

The photographic plate responds only to the intensity which is proportional to the time average of $T^2(x, y, t)$.

Now, the intensity recorded by the photographic plate

$$I(x, y) = \text{Average value of } T^2(x, y, t)$$
$$= \langle T^2(x, y, t)\rangle \quad ...(8.6)$$

or,
$$I(x, y) = A_1^2(x, y)\langle\cos^2(\phi - \omega t)\rangle + A_2^2\langle\cos^2(2\pi\xi x - \omega t)\rangle + 2A_1(x, y)A_2\langle\cos(\phi - \omega t)\cos(2\pi\xi x - \omega t)\rangle$$

We know that

$$\langle\cos^2(\phi - \omega t)\rangle = \frac{1}{2}$$

$$\langle\cos^2(2\pi\xi x - \omega t)\rangle = \frac{1}{2}$$

and $\langle 2\cos(\phi - \omega t)\cos(2\pi\xi x - \omega t)\rangle = \langle\cos(\phi + 2\pi\xi x - 2\omega t) + \cos(\phi - 2\pi\xi x)\rangle$

[Using $2\cos\theta_1\cos\theta_2 = \cos(\theta_1 + \theta_2) + \cos(\theta_1 - \theta_2)$]

The average value of $\cos(\phi + 2\pi\xi x - 2\omega t)$ over the time period T is zero.

So, the intensity $I(x, y)$ is written as

$$I(x, y) = \frac{1}{2}A_1^2(x, y) + \frac{1}{2}A_2^2 + A_1(x, y)A_2\cos(\phi - 2\pi\xi x) \quad ...(8.7)$$

The above equation shows that the intensity is a function of phase $\phi(x, y)$. Thus the phase information of the object wave, which contained in $\phi(x, y)$, is recorded in the intensity pattern. When the photographic plate is developed, one obtains a hologram. The ratio of the transmitted field to the incident field is defined as the transmittance of the hologram that depends on $I(x, y)$. By a suitable developing process one can obtain a condition under which the amplitude transmittance would be linearly related to $I(x, y)$. If $R_e(x, y)$ represents the field of the reconstruction wave, at the hologram plane, then the transmitted field would be

$$T_e(x, y) \propto R_e(x, y)\, I(x, y)$$
$$= C\, R_e(x, y)\, I(x, y) \quad ...(8.8)$$

Where C is the proportionality constant.

Now putting the value of $I(x, y)$ from Eq. (8.7), we have

$$T_e(x, y) = C\, R_e(x, y)\left[\frac{A_1^2(x, y)}{2} + \frac{A_2^2}{2} + A_1(x, y)A_2\cos(\phi - 2\pi\xi x)\right]$$

When the reconstruction wave is identical to the reference wave $R(x, y)$, the above equation becomes

$$T_e(x, y) = C A_2\cos(2\pi\xi x - \omega t)\left[\frac{A_1^2(x, y)}{2} + \frac{A_2^2}{2} + A_1(x, y)A_2\cos(\phi - 2\pi\xi x)\right]$$

$$= CA_2\left[\frac{A_1^2(x,y)}{2}+\frac{A_2^2}{2}\right]\cos(2\pi\xi x-\omega t)+CA_2^2A_1(x,y)\cos(\phi-2\pi\xi x)\cos(2\pi\xi x-\omega t)$$

$$= CA_2\left[\frac{A_1^2(x,y)}{2}+\frac{A_2^2}{2}\right]\cos(2\pi\xi x-\omega t)+\frac{CA_2^2A_1(x,y)}{2}\cos(\phi-\omega t)$$

$$+\frac{CA_2^2A_1(x,y)}{2}\cos(4\pi\xi x-\phi-\omega t) \qquad ...(8.9)$$

[Using $2\cos\theta_1\cos\theta_2=\cos(\theta_1+\theta_2)+\cos(\theta_1-\theta_2)$]

Equation (8.9) gives the transmitted field. The above equation contains three terms. We explain each of the three terms separately.

(i) The first term is nothing but the reconstructed wave whose amplitude is modulated due to the presence of the term $A_1^2(x, y)$. The factor $\cos(2\pi\xi x-\omega t)$ shows that first part of the total field is travelling in the direction of the reference wave.

(ii) The second term represents the original object wave and gives rise to a virtual image.

(iii) The third term is known as the *conjugate object wave*. It contains the phase term $\phi(x, y)$ in addition to the term $4\pi\xi x$, but with a negative sign. It has the reverse curvature to the object wave. Thus if the object wave is a diverging wave then last term represents a converging wave. Thus in contrast to the first term, this wave forms a real image of the object in the space beyond the holographic plate [see Fig. 8.2]

8.6 APPLICATIONS OF HOLOGRAPHY

The principle of holography is used in many fields.

(a) ***Data Storing System***: Holographic data storage is a technique that can store information at high density inside crystals. The advantage of holographic data storage is that the volume of the recording media is used instead of just the surface. If a hologram is recorded of the scene, then events gets frozen into the hologram and hence one can focus through the depth of the reconstructed image and study the phenomenon at later. Holographic storage has the potential to become the next generation of popular storage media.

(b) ***Holographic interferometry***: Holographic interferometry (doubly exposed holography) is a technique that can be used in studying the distribution of strain in an object subjected to stress. The recording plate is first partially exposed to the object wave and reference wave. Then, the object is stressed and the recording plate is again exposed to stressed object wave and reference wave. The recording plate after development forms the hologram. Two object waves emerge from hologram corresponding to stressed and unstressed objects interfere and produce interference fringes. The shape and number of fringes gives the distribution of strain in the object.

(c) ***Holographic scanners***: Holographic scanners are in use in post offices, shipping firms and automated conveyor system to determine the three-dimensional size of a package.

(d) ***Holographic Art***: Holographic art is often the result of collaborations between scientists and artists.

(e) ***Holographic Stereogram***: For making holographic stereograms, photographs of object rather than the real objects are used. The two dimensional photographs are recorded in the form of a composite hologram such that the image appears to be three dimensional.

(f) ***Holographic studios*:** Holographic studios exist in several countries for display holography. These holographic studios are equipped for recording of a wide range of objects and compositions including people and animals.

Now holography is being used in industry, communication and other engineering problems. Another important application is through bar code readers used in shops, libraries warehouses and so on.

Review Exercise

Part 1: Multiple Choice Questions

1. Dennis Gabor received Nobel Prize in Physics for discovering the principles of holography in the year of
   (a) 1948 (b) 1947 (c) 1955 (d) 1971
2. Holography is a method in which one
   (a) records the amplitude only
   (b) records the phase only
   (c) not only records the amplitude but also the phase of the light wave
   (d) none of these
3. The reconstruction process produces
   (a) a virtual image (b) a real image
   (c) both virtual and real image of the object (d) none of these
4. The virtual image is known as
   (a) pseudoscopic image (b) orthoscopic image
   (c) photocopy (d) none of these
5. The real image is known as
   (a) pseudoscopic image (b) photocopy
   (c) orthoscopic image (d) none of these
6. If a hologram breaks into pieces, each piece can reproduce
   (a) the entire image (b) the half image (c) one third image (d) one forth image
7. In the recording of the hologram one superimposes
   (a) object wave and reference wave (b) two object waves
   (c) two reference waves (d) none of these
8. The real image formed
   (a) at the same distance from the hologram, the scene depth is real
   (b) at the same distance from the hologram, the scene depth is inverted
   (c) at the half distance from the hologram, the scene depth is real
   (d) none of these
9. A cylindrical hologram provides a
   (a) 180 degree view of the object (b) 270 degree view of the object.
   (c) 360 degree view of the object (d) 90 degree view of the object

10. For holography system the exposure time is of the order of
    (a) 5 seconds and depends on the colour of the object
    (b) nearly 2 seconds
    (c) 1 second
    (d) 50 seconds

[**Ans.** 1 (d), 2 (c), 3 (c), 4 (b), 5 (a), 6 (a), 7 (a), 8 (b), 9 (c), 10 (a)]

Short Questions with Answers

1. Define holography and hologram.

Ans. Holography is a technique of recording the amplitude and phase of the light waves reflected from an object. Thus, a three dimensional image of the object can be recorded. The recorded photograph is called a hologram.

2. What are object wave and reference wave?

Ans. The light from a laser source is split into two components. One of the component wave is directed towards the object is known as *object wave*. The other one is directed towards the photographic plate is known as *reference wave*.

3. Why is laser needed for holography?

Ans. The coherence length of laser is many metres. Hence holograms of larger object can be produced.

4. Name the principle that is used in the construction and reconstruction of a hologram.

Ans. Development or construction of a hologram is based on the principle of interference of light waves.

Reconstruction of a hologram is based on the principle of diffraction of light waves.

5. What are pseudoscopic and orthoscopic images?

Ans. After reconstruction, the real image formed by hologram is known as pseudoscopic image and virtual image formed by hologram is known as orthoscopic image.

6. What are hologram aberrations?

Ans. Hologram suffer aberrations coursed by a change in the wavelength from construction to reconstruction and also by a mismatch in the reference and reconstruction waves.

7. What are the applications of holograms?

Ans. See Article 8.6

Part 2: Descriptive Questions

1. What is holography? Explain the process of recording and reconstruction of hologram.
2. Explain the theory of holography.
3. What are the basic principle of holography? Why laser needed for holography?
4. Describe holography.
5. State some of the applications of holography.
6. Describe the basic requirements of an ideal display hologram.

CHAPTER 9

Quantum Physics

9.1 INTRODUCTION

The advent of quantum mechanics caused a revolution in the world of physics in the first quarter of the twentieth century. In fact, the formulation of quantum mechanical theory in 1925 was the culmination of researches that had started in the last decade of nineteenth century and came to an end in the third decade of twentieth century. At the end of nineteenth century, physicists had every reason for regarding the laws of newtonian mechanics which governed the motion of material bodies and Maxwell's electromagnetic theory as the fundamental laws of physics. And at that time, there was very little or no reason to suspect the existence of any limitation regarding validity of the aforesaid laws and theories which constitute the branch of physics that is now known as classical mechanics.

But following the discoveries of *X*-rays (by Wilhelm Konrad Röntgen in 1895), radioactivity (by Antoine Henrri Becquerel in 1896), radioactive elements such as radium and polonium (by Pierre Curie and Marie Curie in 1898), Zeeman effect of splitting of the spectral lines in an intense external magnitic field (by Pieter Zeeman in 1896) and electron (by Joshep John Thomson in 1897) set a motion in a series of some important and interesting experiments yielding results which could not be explained with the help of the laws of classical mechanics. In order to resolve the apparent paradoxes which were created by the above-mentioned observations and some other experimental facts, it was necessary to introduce some new ideas that were quite different from the concepts acquired by common sense regarding matter and energy around us and their interaction with each other which (the acquired concepts) were implicit in classical mechanics and had an essential role in its development. And introduction of the aforesaid new ideas caused a revolution in the field of physics which ultimately led to the mathematical formulation of quantum mechanics that had a spectacular as well as immediate success in the explanation of the experimental observation which were the consequences of aforesaid discoveries.

In fact, the researches on discharge of electricity through gases paved the way to conduct experiments which led to the discoveries of *X*-ray, radioactivity, Zeeman effect etc. These discoveries led to the discovery of electron and ultimately helped us to understand the inner structure of the atoms. And hence, we got access to the studies of the atomic and subatomic world. And in this microscopic world, the atomic particles were found to have very high speed comparable to that of light. The behaviours of these particles (with so high speed) could not be explained with the help of the laws of classical mechanics. During the same period, many experiments on black-body radiation were also performed. Physicists like Lummer, Pringsheim and

few others (in 1899) made precise measurements of the intensity distribution of black-body radiation for the entire range of frequencies (or wavelengths). Two physicists, Rayleigh and Jeans in 1900, tried to explain this intensity distribution on the basis of electromagnetic theory of Maxwell but their attempt was partially successful. It was Max Planck who finally explained the said intensity distribution of black-body radiation with the help of his quantum theory of radiation. It was a revolutionary concept not approved by the classical electromagnetic theory. This quantum concept (i.e., particle nature of radiation) was used by Albert Einstein (in 1905) to explain the photoelectric effect. Louis Victor de Broglie (in 1924) predicted the wave nature of the moving microscopic particles of atoms. And this wave nature exhibited by moving particles was experimentally verified by C J Davision and L H Germer of USA (in 1927) and also by G P Thomson of UK independently (in 1928).

Erwin Schrödinger (in 1925) and Werner Heisenberg (in 1926) mainly built up the theoratical framework of quantum mechanics. Subsequently the subject of relativistic quantum mechanics was developed by Paul Adrien Maurice Dirac in 1928. Quantum mechanics was, thus, established as the fundamental theory of the phenomena which occur in the atomic and subatomic world. Though quantum mechanics is a completely new mathematical formulation capable of being applied in the world of microscopic domain, but it also appeared as generalization of classical mechanics. That is the results of quantum mechanics get reduced to those of classical mechanics in the limit when the speed of the particles is relatively small and it is not comparable to that of light in free space.

In the phenomena of macroscopic world the speeds of the particles are comparatively small and the masses of the particles remain practically constant and there is no ambiguity between the nature of particles and that of wave. The spreading of waves along a stretched string or in the water of a still pond is well known to us. While such waves propagate in a medium, energy and momentum are carried from one point to another point along the direction of propagation of the waves. Particles and waves are separate entities in case of classical mechanics. Mechanical waves cause vibration of the particles of medium resulting in the transfer of energy through them. Light is an electromagnetic wave where periodic variation of electric and magnetic fields takes place and this wave also carries energy with itself. There are certain phenomena like interference, diffraction and polarization which are exhibited by waves only. The electromagnetic radiations also exhibit all these aforesaid physical phenomena under suitable conditions. The experimental results in case of some of the experiments involving electromagnetic radiation, which were performed during the last decade of nineteenth century and the first two decades of the twentieth century, could not be explained with the help of the electromagnetic theory. The electromagnetic radiation, in case of these experiments, behaves as a stream of particles and the wave nature of them remains concealed. The electromagnetic wave theory, which explained well the phenomena of interference, diffraction, polarization etc., failed to explain the phenomena of black-body radiation, photoelectric effect, Compton effect etc. where light behaves as a stream of particles. The quantum theory of light was successfully used to explain them. To appreciate the part played by different discoveries related to the development of quantum theory, it is necessary, from the very outset, to have a clear idea of the classical concepts. For this reason, we begin our studies of quantum mechanics with a brief discussion of these concepts, especially black-body radiation and Compton effect.

9.2 BLACK-BODY RADIATION

It is often seen that when some metallic bodies are heated and their temperature is raised to a certain level they start emitting electromagnetic (em) radiation. As an example, let us consider an iron rod. If it is heated to a temperature of a few hundred degrees, it starts giving off em radiation which is predominantly in the

infra-red region. If the temperature is raised to about one thousand degree celsius it begins to glow with reddish colour that means the radiation emitted by it lies in the visible red region having wavelengths shorter than in the previous case. And if it is heated further, it becomes white-hot and the radiation emitted by it gets shifted towards still shorter wavelengths (blue colour) in the visible spectrum of em radiation. Thus, we have seen in the above example that the nature of the radiation depends on the temperature of the emitting body. The heated bodies not only emit radiation but also absorb a part of the radiation to which they are exposed. In 1889 GK Kirchhoff proposed a theorem which states that the ratio of the emissive power of a body to its absorptive power is a constant which depends only on the temperature of the body and not on the nature of it. The emissive power of a body is the amount of radiant energy which is emitted by it per unit area of its surface normally into a unit solid angle* per unit time for a unit range of wavelength. And the absorptive power of a body is the ratio of the amount of radiant energy absorbed by it to the total radiant energy falling on it. So, if a body is able to absorb all the radiant energy which falls on it, then the absorptive power of the body becomes unity. **Such a body which has absorptive power of unity is called a black-body.** Then on the basis of Kirchhoff's theorem as stated above, the emissive power of a black-body is a function of its temperature only since its absorptive power is unity. **A perfect black-body is one which is able to absorb all the radiation of any wavelength completely which is incident on it.** A black-body appears black because no reflection or transmission of light takes place in such a body. **A good absorber of any radiation is also a good emitter of that radiation when it is hot.**

9.2.1 Construction of a Black-Body

In practice, one can realize (construct) an ideal black-body by heating a double-walled hollow metallic enclosure (called cavity) with a small orifice and a conical projection on the inner wall of it which is just opposite to the orifice (Fig. 9.1). The space between the two walls is evacuated to prevent the loss of heat by means of conduction or convection.

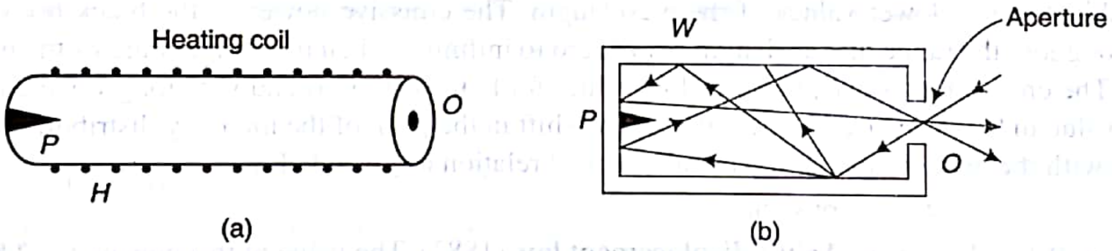


Fig. 9.1 (a) A black-body made of a cylindrical metallic cavity surrounded by a heating coil *H*. (b) The cross-sectional view of the black-body; *O*: the orifice *P*: the conical projection and *W*: the double wall.

The inner surface of the cavity is coated with lampblack which can absorb about 98% of the incident radiant energy. The conical projection prevents those rays to get reflected which enter the cavity through the orifice normally. The radiation which enters the cavity through the orifice is incident on its blackened surface and gets partly absorbed and partly reflected. The component of the incident radiation which gets reflected is incident at another point on the inner surface of the cavity and again gets partly absorbed and partly reflected. This process of absorption and reflection goes on repeatedly since there is very little chance of any part of the said radiation comming out through the orifice again because of its smallness in size. More than 98% of the incident beam is absorbed at each reflection so that ultimately the entire beam of radiation, which enters

*See Appendix A

the cavity, is completely absorbed by it. Thus, the cavity described above has unit absorptive power and it behaves like a perfect black-body.

The inner walls of the heated cavity also emit radiation and a part of the emitted radiation can come out through the orifice. And this emerging radiation has the characteristics of black-body radiation and it is independent of the constructing material of the cavity.

9.2.2 Experimental Observation of Black-body Radiation

By using a black-body as described in the previous subsection and maintaining a constant temperature of it, if one performs an experiment to measure the emissive power (or intensity) of the black-body for different wavelengths, one will obtain the spectra as shown in the Fig. 9.2 for three selected temperatures T_1, T_2 and T_3. For detection of the emitted radiation a bolometer can be used as detector and the spectra can be analyzed by using an infrared spectrometer. Thus the emissive power of the black-body against different wavelengths can be determined.

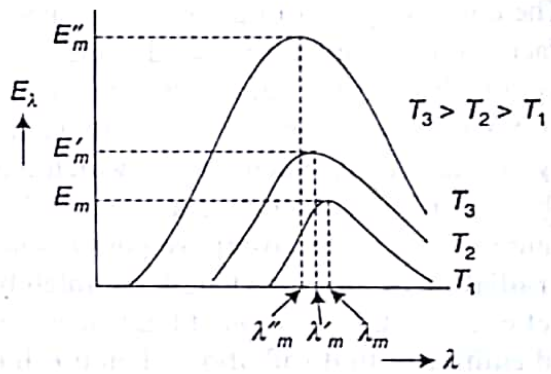


Fig. 9.2 Experimental results of black-body radiation for three temperatures T_1, T_2 and T_3

Having a close look at the figure, one can realize that the spectrum is a continuous curve in which emissive power is a function of wavelength and the wavelength ranges from zero to inifinity. The emissive power E_λ initially increases with respect to wavelength λ, reaching a maximum value at a certain value of the wavelength (λ_m) and then it decreases at the higher values of the wavelength. When temperature increases, the wavelength λ_m corresponding to the highest emissive power is found to shift towards lower values of the wavelength. The emissive power of the black-body radiation increases throughout the range of wavelength (from zero to inifinity) when the temperature of the black-body is increased. The emissive power E_λ has very low values for both very short and very long wavelength. It has a maximum value in between at a wavelength λ_m. The shift in the peak of the intensity distribution curve can be expressed with the help of an empirical mathematical relation as given below:

$$\lambda_m T = \text{constant} \qquad \text{...(9.1)}$$

The relation (9.1) is known as **Wein's displacement law** (1883). The value of this constant is 2.898×10^{-3} mK (metre-kelvin)

One can deduce from thermodynamics a law relating the total emissive power radiated per unit area of a black-body to its absolute temperature. This law is known as **Stefan–Boltzmann law** which can be expressed mathematically as follows:

$$E = \sigma T^4 \qquad \text{...(9.2)}$$

where $E = E(T) = \int_0^\infty E_\lambda(T)\, d\lambda$ and $\sigma = 5.67 \times 10^{-8}$ Wm^{-2} K^{-4}. And σ is known as Stefan's constant. Stefan suggested this law (Eq. (9.2)) from his experimental results in 1879 and Boltzmann theoretically derived it in 1884 from the consideration of thermodynamics. W. Wein (in 1896) from thermodynamical consideration, proposed one empirical relation between E_λ and λ at a constant temperature T which is given by

$$E_\lambda(T) d\lambda = \frac{A}{\lambda^5} f(\lambda T)\, d\lambda \qquad \text{...(9.3)}$$

where A is a constant. And $f(\lambda T)$ is a function of the product of λ and T, i.e., (λT). Equation (9.3) is known as **Wein's distribution law**. Wein's displacement law [Eq. (9.1)] and Stefan–Boltzmann law [Eq. (9.2)] follow simply from Wien's distribution law [Eq. (9.3)]. Using Wein's distribution law [Eq.(9.3)], one can explain the intensity distribution curve for low values of (λT). For higher values of (λT), E_λ comes out to be smaller than the experimental values. So, Wein's distribution law cannot be used, for complete interpretation of experimental data. **Wein's distribution law is also known as Wien's radiation law.**

9.2.3 Theoretical Aspect of Oscillation of Charged Particle

An accelerating charged particle emits e.m. radiation. An oscillaiting charged particle accelerates. So it emits e.m. radiation. Any radiating system is nothing but a collection of charged particles which execute simple harmonic motion through oscillations of its constituent charged particles. The cavity of the black-body as shown in Fig. 9.1 is full of standing waves of e.m. radiation of all wavelengths. When the cavity of the black-body is heated, the atoms in the inner walls of it oscillate with various frequencies and they emit radiations of those frequencies. The energy is continuously exchanged randomly between any two atoms in the cavity walls through emission and absorption of e.m. radiation. The energy density of radiations within the cavity finally reaches an equilibrium condition at a certain constant temperature T. If the temperature increased, then new modes of stationary waves are generated and also the amplitudes of the existing modes increase. The energy density of radiation u_ν at frequency ν increases until a new equilibrium state is reached.

In order to calculate the density of the radiant energy u_ν, i.e., the amount of energy per unit volume, corresponding to radiation with frequencies lying between ν and $\nu - d\nu$ (or ν and $\nu + d\nu$) one can use the following formula

$$u_\nu \, d\nu = n_\nu \, \varepsilon_a \, d\nu \qquad \text{...(9.4)}$$

where, $n_\nu \, d\nu$ is the number of orcillating atoms per unit volume which contribute to the radiations of frequencies lying between ν and $\nu - d\nu$ or (ν and $\nu + d\nu$) and ε_a is the average energy of an oscillating atom at an absolute temperature T of the black-body. Having calculated the number of modes of e.m. stationary waves in a three dimensional box, one can express $n_\nu \, d\nu$ as follows:

$$n_\nu \, d\nu = \frac{8\pi}{c^3} \nu^2 \, d\nu \qquad \text{...(9.5)}$$

where the velocity of light in free space $c = 3 \times 10^8$ m/s.

Having combined Eqs. (9.4) and (9.5), one gets

$$u_\nu \, d\nu = \frac{8\pi}{c^3} \nu^2 \, \varepsilon_a \, d\nu \qquad \text{...(9.6)}$$

Now, one can define a quantity $u_\lambda \, d\lambda$ as the energy density of radiations having wavelengths lying between λ and $\lambda + d\lambda$. Considering the relation between λ and ν one can get

$$c = \nu\lambda$$

or, $$\nu = \frac{c}{\lambda}$$

or, $$d\nu = -\frac{c}{\lambda^2} d\lambda$$

Now, by using these relations $\left(\text{i.e., } c = \nu\lambda \text{ and } d\nu = -\frac{c}{\lambda^2} d\lambda\right)$ in Eq. 9.6, one can get

$$u_\lambda \, d\lambda = \frac{8\pi}{\lambda^4} \varepsilon_a \, d\lambda \qquad ...(9.7)$$

[Since for a particular state of the system the energy density can be denoted by either u_ν or u_λ and $d\lambda = - d\nu$ because while λ increases, ν decrases. The range for $d\lambda$ is between λ and $\lambda + d\lambda$ while the same range for $d\nu$ is between ν and $\nu - d\nu$, hence $u_\lambda \, d\lambda = - u_\nu \, d\nu$]

According to the definition of $u_\nu \, d\nu$ or $u_\lambda \, d\lambda$, $d\nu$ or $d\lambda$ represents infinitesimal change in ν or λ respectively and thus $d\nu$ or $d\lambda$ denotes the range within which we are considering the variable u_ν or u_λ.

One measures the intensity of emitted radiation (E_λ) in any experiment on black-body radiation. The relation between u_λ and E_λ is given by

$$E_\lambda = \frac{c}{4} u_\lambda \qquad ...(9.8)$$

Now, making use of Eqs. (9.7) and (9.8) one can write the following equation:

$$E_\lambda = \frac{2\pi c}{\lambda^4} \varepsilon_a \qquad ...(9.9)$$

This equation [i.e., Eqn. (9.9)] can be used for determination of the expression of intensity.

9.2.4 Rayleigh–Jeans Formula of Radiation

Baron Rayleigh (popularly known as Lord Rayleigh) and J.H. Jeans attempted to derive an expression to express E_λ by making use of Eq. (9.9). They calculated the average energy ε_a of an oscillator on the basis of the equipartition theorem of energy. According to the equipartition theorem, the average energy per degree of freedom of an entity (e.g., molecules of gas, harmonic oscillator etc.) is given by $\frac{1}{2}kT$ where k is Boltzmann's constant and T is the absolute temperature of the concerned system which contains the entity. An e.m. standing wave in the cavity of the black-body originates in an atomic oscillator in the wall of the cavity. As it is a harmonic oscillator, it has two degrees of freedom, one for its kinetic energy and the other for its potential energy. Thus, in thermal equilibrium at an absolute temperature T, the mean energy of each oscillator is given by,

$$\varepsilon_a = kT \qquad ...(9.10)$$

where $k = 1.38 \times 10^{-23}$ Joule/K.

Now, putting the value of ε_a from Eq. (9.10), in Eq. 9.7 and Eq. (9.9), one can get the following two equations:

$$u_\lambda \, d\lambda = \frac{8\pi}{\lambda^4} kT \, d\lambda \qquad ...(9.11)$$

and

$$E_\lambda = \frac{2\pi c}{\lambda^4} kT \qquad ...(9.12)$$

The Eq. (9.11) is known as **Rayleigh–Jeans formula (i.e., law) of radiation** and Eq. (9.12) is a modified form of it, expressed in terms of intensity. The Eq. (9.12), i.e., $E_\lambda = \frac{2\pi c}{\lambda^4} kT$, agrees with the results obtained from experiments in the long wavelength region of the em spectrum. But it fails completely to talley with the experimental results at the shorter wavelength region.

Ultraviolet Catastrophe of Raleigh–Jeans Formula or Jeans' Paradox

Let us consider the modified form of the Rayleign–Jeans' formula given by Eq. (9.12). If one puts $\lambda = 0$ in Eq. (9.12), one gets $E_\lambda = \infty$ i.e., when $\lambda \to 0$, $E_\lambda \to \infty$. But experimentally one gets $E_\lambda = 0$ for $\lambda = 0$, i.e., $E_\lambda \to 0$ when $\lambda \to 0$ (Fig. 9.3).

This discrepancy between Rayleight–Jeans' formula and the experimental results in the shorter wavelengths region is called **ultraviolet catastrophe** or Jeans' paradox. The aforesaid discrepancy between experimental results and theoretical prediction is in fact a failure of the equipartition theorem that is used in the derivation stated above. It is an indication of the limitation of classical mechanics which is based on the equipartition theorem.

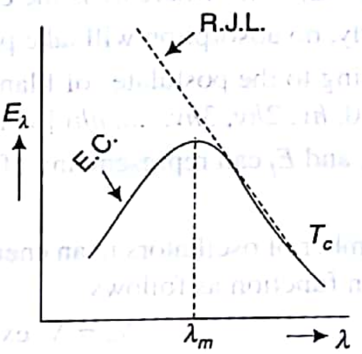


Fig. 9.3 Variation of intensity of black-body radiation against wavelength λ at constant temperature T_c EC: Experimental curve and RJL: the curve on the basis of Rayleigh-Jeans' law.

9.2.5 Planck's Radiation Formula and his Quantum Hypothesis

Max Planck of Germany thought that the failure of Rayleigh–Jeans law may be associated with either equipartition theorem or classical electromagnetic theory or both of them. In 1900, he examined the entire situation critically and came forward with a bold new postulate regarding the nature of the linear harmonic oscillators which remain in a state of equilibrium with the radiant energy within the cavity of the black-body. He stated that an oscillator having given frequency can have energies which are discrete, i.e., the energies can be expressed as an integral multiple of a finite quantum of energy $\varepsilon_o = h\nu$ where h is Planck's constant and ν is the frequency of the oscillator. Thus, the energy of the oscillator can assume such values only which is given by

$$\varepsilon = n\varepsilon_o = nh\nu$$

where n is an integer including zero. This statement is known as **Planck's quantum hypothesis.** Obviously the energy of the lowest state (i.e., the ground state) of the oscillator will be zero.

It was also assumed by Planck that the change in the amount of energy of the oscillator due to absorption or emission of radiant energy can also take place by a discreate amount $h\nu$. He calculated the value of the constant h by fitting his theory to the experimental data (Fig. 9.4).

The value of the Planck's constant is given by

$$h = 6.626 \times 10^{-34} \text{ joules-second.}$$

It is a universal constant and it has crucial role to play in all quantum phenomena.

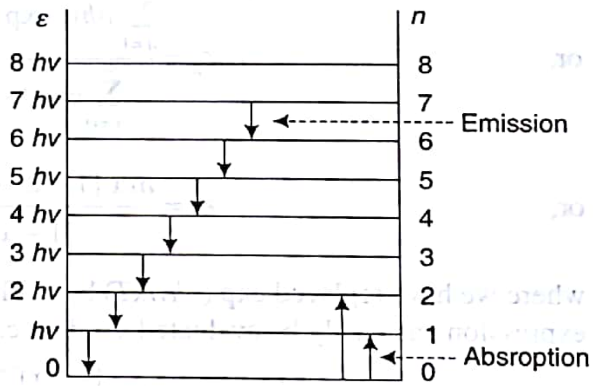


Fig. 9.4 Energy levels of an harmonic oscillator $\uparrow \Rightarrow$ absorption and $\downarrow \Rightarrow$ emission of energy.

Since according to Max Planck, the change in the energy of an oscillator can only take place in discrete form, the amount of energy emitted by the oscillator or absorbed by it in a single step of change is called a photon and its value will be $h\nu$ which is equal to the loss of energy in case of emission and gain of energy in case of absorption.

No emission of energy from the oscillator will take place unless the change in energy of the oscillator $\Delta E = -(E_f - E_i) = h\nu$ where $h\nu$ is the energy carried by the single emitted photon.

Similarly, no absorption will take place unless the energy of the absorbed photon $h\nu = E_f - E_i = \Delta E$.

According to the postulates of Planck, an oscillator can exist only in a set of states with discrete energies given by $\{0, h\nu, 2h\nu, 3h\nu, ..., nh\nu\}$ (Fig. 9.4)

Here E_i and E_f can represent any of the values of the set of discrete energies of the oscillators given by,

$$\{0, h\nu, 2h\nu, 3h\nu, ..., nh\nu\}$$

The number of oscillators in an energy level ε_ν $(= nh\nu)$ is always determined by the Maxwell–Boltzmann's distribution function as follows:

$$N_n = N_o \exp(-\varepsilon_\nu/(kT))$$

or,

$$N_n = N_o \exp(-nh\nu/(kT))$$

$N_n = N_o$ for $n = 0$, i.e., the number oscillators in the ground state is given by N_o. The number of oscillators in the nth state is N_n and it decreases exponentially with increasing energy ε_ν of the state.

Let us first derive an expression for the average energy ε_a of the oscillators on the basis of the new ideas as stated above and given by Max Planck. Since the energies of the oscillators can assume only discrete values, accordingly we have to replace integral symbols in the expression by summation symbols over all possible oscillar states in order to determine the average energy ε_a.

Now, we can express ε_a as follows:

$$\varepsilon_a = \frac{\sum_{n=0}^{\infty} N_n \varepsilon_n}{\sum_{n=0}^{\infty} N_n} = \frac{\sum_{n=0}^{\infty} N_o \varepsilon_n \exp(-\varepsilon_n/(kT))}{\sum_{n=0}^{\infty} N_o \exp\left(\frac{-\varepsilon_n}{kT}\right)}$$

or,

$$\varepsilon_a = \frac{\sum_{n=0}^{\infty} nh\nu . \exp(-nh\nu/(kT))}{\sum_{n=0}^{\infty} \exp(-nh\nu/(kT))}$$

or,

$$\varepsilon_a = \frac{h\nu x\,[1 + 2x + 3x^2 + 4x^3 + ... + \infty]}{[1 + x + x^2 + ... + \infty]}$$

where we have replaced $\exp(-h\nu/kT)$ by x. The sums in both the numerator and the denominator of the above expression can easily be evaluated. And on calculation of the sums as mentioned above, we obtain

$$\varepsilon_a = h\nu x \frac{(1-x)^{-2}}{(1-x)^{-1}} = \frac{h\nu x}{1-x} = \frac{h\nu}{\frac{1}{x} - 1}$$

Hence, we can write

$$\varepsilon_a = \frac{h\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1} \qquad ...(9.13)$$

One can easily show that if $h\nu \ll kT$, then the value of ε_a reduces to the classical limit kT. This obviously corresponds to the case of continuous variation of the oscillator energy. Equation (9.13) along with Eq. (9.7) gives Planck's radiation formula as follows:

$$u_\lambda\, d\lambda = \frac{8\pi}{\lambda^4} \varepsilon_a\, d\lambda$$

or, $$u_\lambda\, d\lambda = \frac{8\pi}{\lambda^4} \left(\frac{h\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1} \right) d\lambda$$

or, $$u_\lambda\, d\lambda = \frac{8\pi}{\lambda^4} \left(\frac{h \times \frac{c}{\lambda}}{\exp\left(\frac{hc}{\lambda kT}\right) - 1} \right) d\lambda \quad [\because c = \nu\lambda]$$

or, $$u_\lambda\, d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{\exp\left(\frac{hc}{\lambda kT}\right) - 1} d\lambda \qquad \text{...(9.14)}$$

Equation (9.14) is known as *Planck's radiation formula* (i.e., law) for the intensity distribution of black-body radiation. It agrees very well with the experimental results, both for the long and the short wavelength ends of the energy spectrum.

Let us discuss two special cases of Planck's radiation law wherein the wavelength assumes two extreme values given by $\lambda = 0$ and $\lambda = \infty$.

Case 1 When $\lambda \to \infty$, $\frac{hc}{\lambda kT} \to 0$. Equation (9.14) can be written as

$$u_\lambda\, d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{1 + \frac{hc}{\lambda kT} + \left(\frac{hc}{\lambda kT}\right)^2 + \ldots - 1} d\lambda$$

Now, we can neglect the higher powers of $\left(\frac{hc}{\lambda kT}\right)$ as $\frac{hc}{\lambda kT} \to 0$. So the above equation will take the following form:

$$u_\lambda\, d\lambda = \frac{8\pi}{\pi^4} kT\, d\lambda$$

or, $$u_\lambda = \frac{8\pi}{\lambda^4} kT$$

or, $$E_\lambda = \frac{c}{4} u_\lambda = \frac{c}{4} \times \frac{8\pi}{\lambda^4} kT$$

i.e., $$E_\lambda = \frac{2\pi c}{\lambda^4} kT$$

This is same as the expression obtained from Rayleigh–Jeans' theory [vide Eq. (9.12)].

Case 2 When $\lambda \to 0$, $\exp\left(\frac{hc}{\lambda kT}\right) >> 1$.

Hence, from Eq. (9.14), we can obtain,

$$u_\lambda\, d\lambda = \frac{8\pi hc}{\lambda^5} \exp\left(-\frac{hc}{\lambda kT}\right) d\lambda$$

or, $$E_\lambda = \frac{c}{4} u_\lambda = \frac{c}{4} \times \frac{8\pi hc}{\lambda^5} \exp\left(-\frac{hc}{\lambda kT}\right)$$

or, $$E_\lambda = \frac{2\pi hc^2}{\lambda^5} \exp\left(-\frac{hc}{\lambda kT}\right)$$

This is identical to the Eq. (9.3) which is the Wein's empirical formula for intensity distribution of radiation and $f(\lambda T)$ is an exponential function.

Thus, it has been seen that in the limiting cases of extremely low and high wavelengths, Planck's radiation law reduces well to Rayleigh–Jeans law and Wein's distribution law (also known as Wein's radiation law) respectively.

9.2.6 Deductions from Planck's Radiation Formula

(a) Wien's Displacement Law Planck's radiation formula is given by

$$u_\lambda \, d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda kT}\right) - 1} d\lambda$$

or, $$E_\lambda = \frac{c}{4} u_\lambda = \frac{2\pi hc^2}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda kT}\right) - 1}$$

or, $$E_\lambda = \frac{2\pi hc^2}{z}$$

where, $z = \lambda^5 \left\{\exp\left(\frac{hc}{\lambda kT}\right) - 1\right\}$

Now, z will be minimum if $\frac{dz}{d\lambda} = 0$

or, if $$\frac{dz}{d\lambda} = 5\,\lambda^4 \left\{\exp\left(\frac{hc}{\lambda kT}\right) - 1\right\} - (\lambda^5) \frac{hc}{\lambda^2 kT} \exp\left(\frac{hc}{\lambda kT}\right) = 0$$

or, if $$(5\lambda) \left\{\exp\left(\frac{hc}{\lambda kT}\right) - 1\right\} = \frac{hc}{kT} \exp\left(\frac{hc}{\lambda kT}\right)$$

or, if $$1 - \exp\left(-\frac{hc}{\lambda kT}\right) = \frac{hc}{5\lambda kT}$$

or, if $$1 - \exp(-x) = \frac{x}{5} \quad \text{where } x = \frac{hc}{\lambda kT}$$

Now, if we denote the value of λ for which the value of the function z becomes minimum by λ_m, then we can arrange the condition for minimum value of z as follows:

The function z will be minimum if $\{1 - \exp(-x)\} = \frac{x}{5}$ when $x = \frac{hc}{\lambda_m kT}$

The equation

$$1 - \exp(-x) = \frac{x}{5} \qquad ...(9.15)$$

is a transcendental equation which cannot be solved analytically. One can solve it easily by using the graphical method.

If one puts

$$y = 1 - \exp(-x) \text{ and } y = \frac{x}{5}$$

Then the point of intersection of the two graphs corresponding to the above two equations gives the solution of Eq. (9.15). And the solution turns to be $x = 4.956$ (Fig. 9.5). One can easily verify correctness of this solution by putting this value of x in Eq. (9.15).

We thus can have the value of the product $(\lambda_m T)$ as follows:

$$\lambda_m T = \frac{hc}{kx} \left[\because x = \frac{hc}{\lambda_m kT} \right]$$

or, $\lambda_m T = \dfrac{hc}{4.965\,k}$ [Putting $x = 4.965$]

or, $\lambda_m T = 0.29 \times 10^{-2}$ m

(Putting values of h, c and k)

or, $\lambda_m T = \text{constant} = c'$ (say).

or, $\lambda_m T = c'$ ...(9.16)

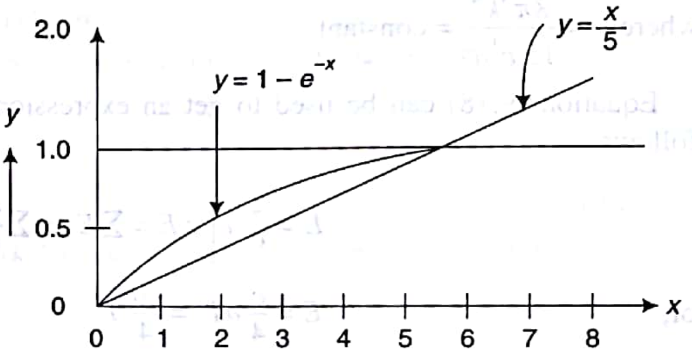


Fig. 9.5 **Graphical solution of the equations $y = \frac{x}{5}$ and $y = 1 - e^{-x}$.**

The Eq. (9.16) is **Wien's displacement law.**

(b) Stefan–Boltzmann Law The total energy density u of the radiation is given by

$$u = \int_0^\infty u_\nu\, d\nu = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3\, d\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1} \quad ...(9.17)$$

$\left[\text{by using Eq. (9.14) and } \lambda = \frac{c}{\nu}\right]$

If one substitutes $x = \dfrac{h\nu}{kT}$, one can have $dx = h\,d\nu/(kT)$, so that

$$u = \frac{8\pi h}{c^3} \left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3\, dx}{\exp(x) - 1}$$

Now, to evaluate the above definite integral (on the right hand side of the above equation) having lower limit 0 and upper limit ∞, one can note that

$$\int_0^\infty \frac{x^3\, dx}{\exp(x) - 1} = \int_0^\infty x^3 \{\exp(-x) + \exp(-2x) + \exp(-3x) + \ldots\}\, dx$$

$$= \sum_p \int_0^\infty x^3 \cdot \exp(-px)\, dx = \sum_p \frac{3!}{p^4}$$

since $\displaystyle\int_0^\infty x^n \exp(-px)\, dx = \frac{n!}{p^n}$

Also $$\sum_p \frac{1}{p^4} = \frac{\pi^4}{90}, \text{ so that}$$

one gets

$$u = \frac{8\pi k^4 T^4}{c^3 h^3} \frac{\pi^4}{15} = aT^4$$

i.e., $$u = aT^4 \quad \ldots(9.18)$$

where $a = \dfrac{8\pi^5 k^4}{15\, c^3 h^3}$ = constant.

Equation (9.18) can be used to get an expression between intensity E and absolute temperature T as follows:

$$E = \frac{4}{c} u \left[\because E = \sum_\lambda E_\lambda = \sum_\lambda \frac{4}{c} u_\lambda = \frac{4}{c} u\right]$$

or, $$E = \frac{c}{4} aT^4 = \frac{ca}{4} T^4$$

or, $$E = 6\, T^4 \quad \ldots(9.19)$$

where the Stefan's constant σ is given by

$$\sigma = \frac{ca}{4} = \frac{2\pi^5}{15} \frac{k^4}{c^2 h^3} = 5.67 \text{ Wm}^{-2} \text{ K}^{-4}$$

Equation (9.19) is **Stefan's law.**

9.2.7 Einstein's Concept Regarding Black-body Radiation

Planck's radiation formula can be derived alternatively on the basis of a technique based on quantum statistics. For detailed discussion of the technique, one can refer to any text book on statistical mechanics. According to Planck's quantum hypothesis, the oscillators emitting em radiations have energies given by the relation $\varepsilon = n\varepsilon_o = nh\nu$ (n = 0, 1, 2, 3, 4, ...). And according to Einstein's assumption, the em field itself is quantized and light consists of energy packets or particles, known as photons, each of which travels with a velocity of $c = 3 \times 10^8$ m/s in free space. The energy carried by each photon is given by

$$E = h\nu \quad \ldots(9.20)$$

or, $$E = \frac{hc}{\lambda} \quad \ldots(9.21)$$

Thus, one can consider the black-body radiation as a photon-gas in thermal equilibrium with the atoms in the walls of the black-body cavity. And using this idea, one can derive Planck's radiation formula with the help of the techniques of qunatum statistical mechanics.

9.3 RELATIVISTIC PROPERTIES OF PARTICLES WITH HIGH SPEED COMPARABLE TO THAT OF LIGHT

If a particle with rest mass m_0 is moving with a velocity v which is comparable to that of light in free space c, then the mass of the particle will no longer be constant; in fact, it will vary following the formula given below:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{...(9.22)}$$

where m is the mass of the particle while it moves with speed v which is comparable with c.

According to Einstein's special theory of relativity the mass of a particle can be converted into equivalent energy which is given by the following equation:

$$\Delta E = \Delta mc^2 \quad (\Rightarrow E = mc^2) \qquad \text{...(9.23)}$$

This equation is known as mass-energy equivaence relation.

By using the concepts of special theory of relativity, one can derive a relation between rest mass m_0 of a body and its momentum as given below:

$$E^2 = p^2c^2 + m^2c^4$$

$$E = \sqrt{p^2c^2 + m^2c^4} \qquad \text{...(9.24)}$$

Interested readers may refer to Appendix-C (Relativistic Concept) for the derivation of equations (9.22), (9.23) and (9.24).

9.4 COMPTON EFFECT

The picture of Max Planck and Albert Einstein regarding individual energy packages (i.e., light quanta) proves only that the exchange of energy between atoms and light beams takes place by quanta. But there was no conclusive experimental evidence available in support of particle nature of light photons till 1921. In the year of 1922, Arthur Holly Compton of USA discovered an effect which was later named after him and **this discovery made one experimental evidence available in support of corpuscular behaviour of light photons with only one exception that photons have restmass equal to zero. This is because light photons never remain at rest.** Compton liked to visualize the collisions between photons and electrons as similar to those between two ivory balls on a billiard table. And for such scattering to happen, the electron should be a free electron. But this is almost impossible in case of an experimental arrangement.

AH Compton knew the fact that the binding energy of the electrons of outermost shell of an atom is comparable in magnitude to the energy of photons of visible light which ranges from 400 nm to 700 nm. In order to make the impact between the light photon and the electron overpoweringly strong (so that the electron can be considered as almost free) he thus selected for his experiments high-frequency X-rays, which are energy-rich photons having much more energy. He found (through his experiments with X-rays) that the scattered X-ray emerged with a somewhat longer wavelength (or shorter frequency) than that of incident X-rays. **This change in the wavelength of the colliding photon is called Compton shift and the phenomenon of wavelength shift of a photon during the collision of the same with a matter particle is called Compton effect.** On the basis of electromagnetic wave theory this phenomenon cannot be explained because according to this theory a wave could only shake the electron in the atoms of the target with a frequency equal to that of the incident wave but it cannot change its wavelength.

9.4.1 Derivation of Compton Formula

The result of the said Compton effect, however, could be explained at once if, (by taking the corpuscular point of view into consideration), one regards the process as one elastic collision of two particles, the electron and a X-ray photon as shown in Fig. 9.6.

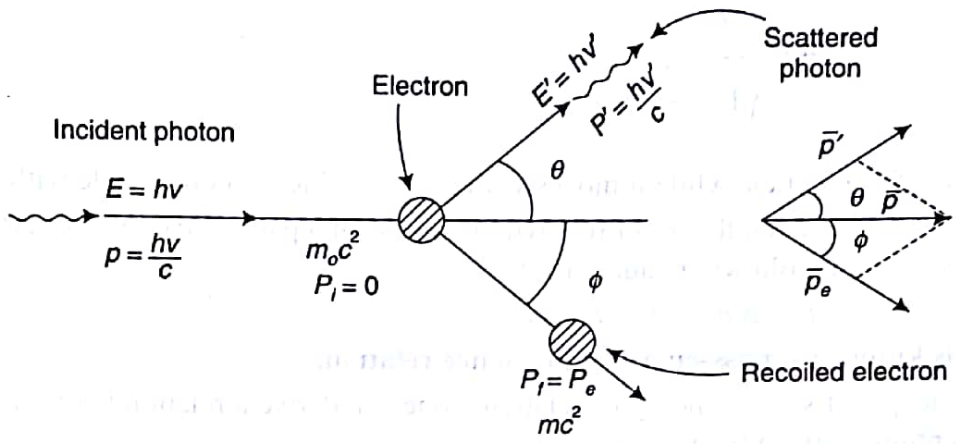


Fig. 9.6 Compton effect scattering of an electron due to collision with a photon

The calculations are made on the basis of the laws of conservation of momentum and energy. The energy possessed by the light photon before collision is $h\nu$ and the momentum possessed by it is $h\nu/c$. When the light photon of energy $h\nu$ strikes the electron at rest, it imparts kinetic energy to the electron and therefore itself looses some energy. The scattered photon possesses less amount of energy $h\nu'$ and momentum $h\nu'/c$. For the sake of simplicity it was assumed that the electron is at rest. So in such a case its energy is the rest mass energy, m_oc^2, while its momentum is zero. If v be the velocity of the electron after collision, then its mass m at this speed v, is given by Einstein's formula, i.e., by the following equation,

$$m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{...(9.25)}$$

Before the collision the energy of the electron is m_oc^2 and after the collision its energy is mc^2. So its kinetic energy (T_e) after the collision is given by

$$T_e = mc^2 - m_oc^2 = \frac{m_oc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_oc^2$$

or,
$$T_e = (m - m_o)\,c^2 = m_oc^2\left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1\right)$$

or,
$$T_e = (m - m_o)\,c^2 = m_oc^2\left(\frac{1}{\sqrt{1 - \beta^2}} - 1\right) \qquad \text{...(9.26)}$$

where $\beta = \frac{v}{c}$

As can be seen in the Fig. 9.6, θ is the angle of deviation of the photon, and ϕ is angle of recoil of the electron following the collision.

By using the law of conservation of energy we can write

$$h\nu + m_oc^2 = h\nu' + mc^2 \qquad \text{...(9.27)}$$

Again by using the law of conservation of linear momentum and also using the fact that its component along any given direction is conserved, we can write the following two equations:

$$\frac{h\nu}{c} = \frac{h\nu'}{c}\cos\theta + mv\cos\phi$$

and
$$0 = \frac{h\nu'}{c}\sin\theta - mv\sin\phi$$

These two equations can be rewritten as follows

$$\frac{h\nu}{c} = \frac{h\nu'}{c}\cos\theta + \frac{m_o\beta c\cos\phi}{\sqrt{1-\beta^2}} \quad ...(9.28)$$

and
$$0 = \frac{h\nu'}{c}\sin\theta - \frac{m_o\beta c}{\sqrt{1-\beta^2}}\sin\phi \quad ...(9.29)$$

$$\left[\because \beta = \frac{v}{c} \Rightarrow v = \beta c\right]$$

Now, Eq. (9.28) can be rewritten as

$$\frac{m_o\beta c}{\sqrt{1-\beta^2}}\cos\phi = \frac{h\nu}{c} - \frac{h\nu'}{c}\cos\theta \quad ...(9.30)$$

and Eq. (9.29) can be rewritten as

$$\frac{m_o\beta c}{\sqrt{1-\beta^2}}\sin\phi = \frac{h\nu'}{c}\sin\theta \quad ...(9.31)$$

Squaring and adding Eq. (9.30) and Eq. (9.31), we get

$$\frac{m_o^2\beta^2c^2}{(1-\beta^2)} = \left(\frac{h\nu}{c}\right)^2 + \left(\frac{h\nu'}{c}\right)^2 - \frac{2h^2\nu\nu'}{c^2}\cos^2\theta \quad ...(9.32)$$

Equation (9.27) can be rearranged as

$$mc^2 = h\nu - h\nu' + m_oc^2$$

or,
$$\frac{m_oc^2}{\sqrt{1-\beta^2}} = h\nu - h\nu' + m_oc^2$$

Now by squaring both sides, we can write

$$\frac{m_o^2c^4}{(1-\beta^2)} = (h\nu - h\nu' + m_oc^2)^2$$

or,
$$\frac{m_o^2c^2}{(1-\beta^2)} = \left(\frac{h\nu}{c} - \frac{h\nu'}{c} + m_oc\right)^2$$

or,
$$\frac{m_o^2c^2}{(1-\beta^2)} = \left(\frac{h\nu}{c}\right)^2 + \left(\frac{h\nu'}{c}\right)^2 - \frac{2h^2\nu\nu'}{c^2} + m_o^2c^2 + 2(m_oh)(\nu - \nu') \quad ...(9.33)$$

Now subtracting Eq. (9.32) from Eq. (9.33), we obtain

$$\frac{m_o c^2}{(1-\beta^2)} \times (1-\beta^2) = m_o^2 c^2 - \frac{2h^2 vv'}{c^2}(1-\cos\theta) + 2 m_o h (v - v')$$

or, $$2m_o h (v - v') = \frac{2h^2 vv'}{c^2}(1-\cos\theta)$$

or, $$m_o hc\left(\frac{v-v'}{vv'}\right) = \frac{h^2}{c}(1-\cos\theta)$$

or, $$c\left(\frac{1}{v'}-\frac{1}{v}\right) = \frac{h}{m_o c}(1-\cos\theta)$$

or, $$\frac{c}{v'}-\frac{c}{v} = \frac{h}{m_o c}(1-\cos\theta)$$

or $$\lambda' - \lambda = \frac{h}{m_o c}(1-\cos\theta) \quad [\because v\lambda = c]$$

or, $$\Delta\lambda = \frac{h}{m_o c}(1-\cos\theta), \quad \text{where } \Delta\lambda = \lambda' - \lambda$$

or, $$\Delta\lambda = \lambda_c(1-\cos\theta), \quad \left[\text{Putting } \lambda_c = \frac{\lambda}{m_o c}\right]$$

or, $$\Delta\lambda = 2\,\lambda_c \sin^2\frac{\theta}{2} \qquad \text{...(9.34)}$$

where Compton wavelength,

$$\lambda_c = \frac{h}{m_o c} = 2.42 \times 10^{-3}\ \text{nm}.$$

Equation (9.34) is known as **Compton formula** for the change of wavelength of the photon due to the scattering process. **The parameter $\lambda_c \left(= \frac{h}{m_o c}\right)$ is called the Compton wavelength which is a universal constant;** λ and λ' are respectively the wavelengths of the incident and scattered photons. Equation (9.34) expresses the wavelength shift due to Compton scattering in terms of Compton wavelength λ_c and scattering angle of the photon θ. **The wavelength shift is also known as Compton shift and it is independent of the wavelength of incident radiation. It is also independent of the scattering material.** For a scattering angle of $\pi/2$,

$$\Delta\lambda = 2\lambda_c \sin^2\frac{\pi}{4} = \lambda_c$$

The above results were verified experimentally by Compton himself by using *X*-rays of different wavelengths. Gamma radiations (γ- rays) also exhibit Compton effect.

9.4.2 Derivation of the Angle of Recoil of the Electron

The kinetic energy of the recoiled electron T_e can easily be calculated.

Now, dividing Eq. (9.31) by Eq. (9.30) one gets.

$$\tan\phi = \frac{\nu'\sin\theta}{\nu - \nu'\cos\theta} = \frac{\sin\theta}{\frac{\nu}{\nu'} - \cos\theta} \qquad \text{...(9.35)}$$

In Eq. (9.34), $\Delta\lambda$ is expressed as

$$\Delta\lambda = 2\lambda_c \sin^2\frac{\theta}{2}$$

or, $$\lambda' - \lambda = 2\lambda_c \sin^2\frac{\theta}{2}$$

or, $$\frac{c}{\nu'} - \frac{c}{\nu} = \frac{2h}{mc}\sin^2\frac{\theta}{2}$$

or, $$\frac{1}{\nu'} - \frac{1}{\nu} = \frac{2h}{m_o c^2}\sin\frac{\theta}{2}$$

Now, putting $\alpha = h\nu/(m_o c^2)$ or $\frac{\alpha}{\nu} = h/(m_o c^2)$ in the above equation, we get

$$\frac{1}{\nu'} - \frac{1}{\nu} = \frac{2\alpha}{\nu}\sin^2\frac{\theta}{2}$$

or, $$\frac{1}{\nu'} = \frac{1}{\nu} + \frac{2\alpha}{\nu}\sin^2\frac{\theta}{2}$$

or, $$\frac{1}{\nu'} = \frac{1}{\nu}\left(1 + 2\alpha\sin^2\frac{\theta}{2}\right)$$

or, $$\nu' = \frac{\nu}{1 + 2\alpha\sin^2\frac{\theta}{2}} \qquad \text{...(9.36)}$$

Now, substituting the value of ν' from Eq. (9.36) in the Eq. (9.35) one can get,

$$\tan\phi = \frac{\sin\theta}{1 + 2\alpha\sin^2\frac{\theta}{2} - \cos\theta}$$

or, $$\tan\phi = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{(1-\cos\theta) + 2\alpha\sin^2\frac{\theta}{2}}$$

or, $$\tan\phi = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2} + 2\,\alpha\sin^2\frac{\theta}{2}}$$

or, $$\tan\phi = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}(1+\alpha)}$$

or, $$\cot \phi = (1 + \alpha) \tan \frac{\theta}{2} \quad ...(9.37)$$

9.4.3 Derivation of Kinetic Energy of the Electron

Equation (9.37) gives the relationship between the angle of scattering of the photon (θ) and the angle of recoil of the electron (ϕ). So by detecting the angle of scattering of the photon, one can calculate the angle of recoil of the electron.

The kinetic energy of the recoiled electron can be calculated as follows:

$$T_e = mc^2 - m_o c^2 = h\nu - h\nu'$$

or, $$T_e = h\left[\nu - \frac{\nu}{1 + 2\alpha \sin^2 \frac{\theta}{2}}\right], \quad \text{using Eq. (9.36)}$$

or, $$T_e = h\nu\left[1 - \frac{1}{1 + 2\alpha \sin^2 \frac{\theta}{2}}\right]$$

or, $$T_e = h\nu \cdot \frac{\alpha(1 - \cos \theta)}{1 + \alpha(1 - \cos \theta)} \quad ...(9.38)$$

So, if one knows the frequency of the incident photon and angle of deviation of scattered photon, then one can use Eq. (9.38) to calculate the kinetic energy of the recoiled electron.

9.4.4 Experimental Set-up for Study of Compton Effect

In Fig. 9.7, a typical experimental set up for the study of Compton effect is shown.

In the X-ray generating tube T is a molybdenum target. The monochromatic k_α radiation from the target T is incident on the carbon scatterer S. The beam of X-ray is scattered through an angle θ and then passes through the collimator C. Ultimately the scattered beam falls on a crystal C_r or in a Bragg spectrometer. The X-rays are diffracted by C_r and then enter the detector D which is basically an ionization chamber. The detector D measures the intensity of the diffracted beam. The X-rays interact with the loosely bound electrons of the carbon scatterer S and get deviated through an angle θ (see Fig. 9.6) and the electrons get recoiled through angle ϕ (not shown in Fig. 9.7). At the Bragg spectrometer C_r, the X-rays get diffracted through various angles depending on the order of diffraction. The angle δ is one such diffraction angle. By measuring the angles of diffraction at which the maxima of intensity are observed, one can determine the wavelength of the X-rays scattered by S at a given angle of diffracton δ from Bragg's law given by

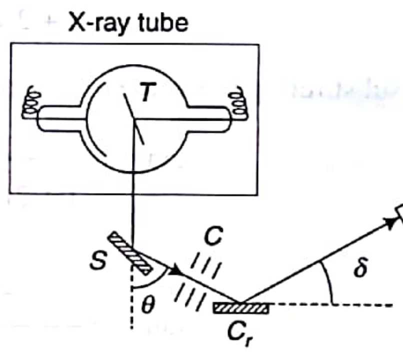


Fig. 9.7 Experimental set-up for the study of Compton effect.

$$n\lambda = 2\, d \sin \delta \quad ...(9.39)$$

For the derivation of Eq. (9.39), the reader is referred to Chapter 10. The intensity of diffracted X-rays were measured as a function of wavelength at different angle of deviation θ (see Fig. 9.6). For each angle of deviation θ (other than $\theta = 0°$) two peaks appear corresponding to the deviated X-ray photons with two different wavelengths. The intensity I versus wavelength λ curves are shown for different angles of deviation in Fig. 9.8.

The first peak represents the wavelength of the incident beam and the second one represents a wavelength which is greater than the incident wavelength. The difference in the two wavelengths is greater for a greater angle θ.

At $\theta = 90°$, one of the peaks was found to have wavelength of molybdenum k_α X-rays (i.e., 7.08×10^{-2} nm) while the other had the wavelength of 7.316×10^{-2} nm. The wavelength difference 0.236×10^{-2} nm between the two wavelengths agrees very well with the value of $\Delta\lambda$ as expected from the Compton equation (i.e. Eq. (9.34)) for $\theta = 90°$ ($\Delta\lambda = 0.236 \times 10^{-2}$ nm).

The wavelength difference was calculated by him having used monochromatic X-rays of different wavelengths. And $\Delta\lambda$ was found to possess the same value in all cases.

Results of observation at other angles of deviation had also confirmed the Compton's theoretical predictions.

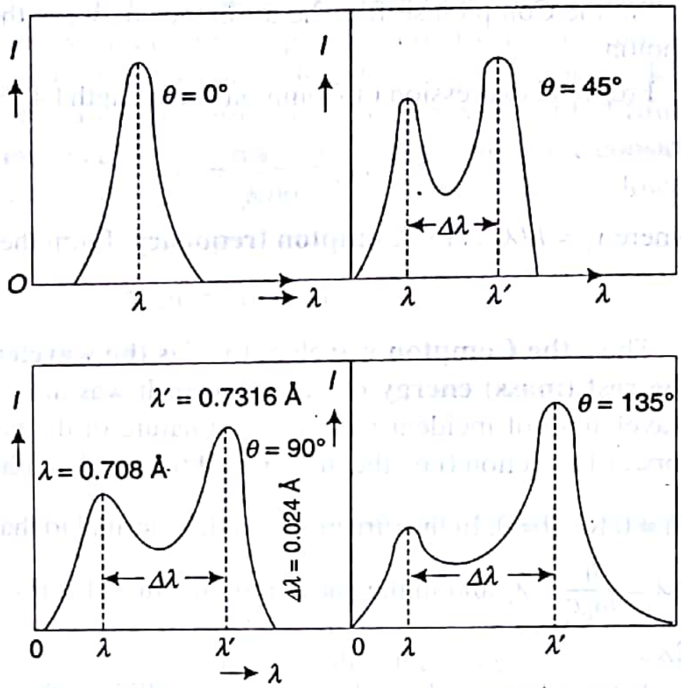


Fig. 9.8 Compton shift ($\Delta\lambda$) at four angles of scattering.

One can note here that the presence of the peak at longer wavelength (7.316×10^{-2} nm for $\theta = 90°$) is due to Compton scattering from the electron that may be considered free as its energy of binding with the atom is small enough as compared to that of the X-ray photon which is $h\nu$. On the other hand, the presence of the other peak at the wavelength of the incident X-ray is due to scattering from one bound electron in which case the recoiled momentum is taken up by the entire atomic mass. As the mass of the atom is much heavier as compared to that of the electron, a negligible wavelength shift is produced in this case $\left(\text{because } \Delta\lambda = (2h/(m_o' c^2)) \sin^2 \frac{\theta}{2} \text{ and } m_o' \text{ is the rest mass of the atom}\right)$. We get conclusive evidence in support of the particle character of electromagnetic radiation from the Compton effect. On analysis of Eq. (9.34), i.e.,

$$\cot \phi = (1 + \alpha) \tan \frac{\theta}{2}$$

We see that as the angle of scattering (deviation) of the photon θ varies between 0 and π, the angle of recoil of the electron ϕ varies from $\frac{\pi}{2}$ to 0, i.e., the electrons are ejected at angles less than $\frac{\pi}{2}$. T_e is maximum for $\phi = 0$ and minimum for $\phi = \frac{\pi}{2}$, i.e., for $\theta = \pi$ and $\theta = 0$ respectively. As discussed above (earlier), the Compton (wavelength) shift is given by

$$\Delta\lambda = \frac{h}{m_o c}(1 - \cos\theta)$$

or,
$$\Delta\lambda = \lambda_c (1 - \cos\theta)$$

where $\lambda_c = \dfrac{h}{m_o c}$

So, the Compton shift is the difference between the wavelengths of the incident phóton and the scattered photon.

From the expression of Compton wavelength $\left(\lambda_c = \frac{h}{m_o c}\right)$ one can write the following equation:

$$m_o c^2 = \frac{hc}{\lambda_c} = h\nu_c$$

where $\nu_c = h/\lambda_c$ is the **Compton frequency**. From the above equation one can write

$$E_c = h\nu_c = m_o c^2$$

Thus, **the Compton wavelength λ_c is the wavelength of radiation whose photon has energy equal to the rest (mass) energy of an electron.** It was already noted that Compton shift $\Delta\lambda$ is independent of the wavelength of incident radiation and nature of the target. It depends only on the angle of scattering. In the forward direction (i.e., the direction of the incident photon where $\theta = 0$), the Compton shift is zero as $(1 - \cos\theta) = 0$ for $\theta = 0$. In the direction which is normal to that of the incident photon where $\theta = \frac{\pi}{2}$, the Compton shift $\Delta\lambda = \frac{h}{m_o c} = \lambda_c$ and in the backward direction (for $\theta = \pi$) the Compton shift is the maximum and it is equal to $2\lambda_c$.

When photons of incident radiation collide with tightly bound electrons of the target atom, the energy and momentum of the colliding photon are exchanged with the atom as a whole. Since the mass of the target atom is much greater than that of a free electron (or lightly bound electron of metal atom), the Compton shift in this case is neglected, and λ' practically coincides with λ. This explains the presence of the unmodified radiation at all angles of scattering of the incident radiation (vide Fig. 9.8).

9.4.5 Failure of Classical Theory in Case of Compton Effect

According to classical wave theory X-rays are electromagnetic waves. If ν be the frequency of the incident X-ray radiation which falls on the material, then they cause electrons in the material to oscillate at the same frequency ν. The oscillating electrons then radiate X-rays of equal frequency ν. Hence the scattered X-rays should consist of single frequency ν. Also, the electrons should radiate X-ray waves uniformly in all possible directions. Thus, the wavelength of the scattered X-ray radiation should not vary with the angle of scattering θ. But this is contrary to the experimental fact which was revealed by Compton. So, classical theory fails to explain Compton effect.

9.5 WAVE PARTICLE DUALITY

It was earlier observed that radiations very well exhibit the well-known classical phenomena of interference, diffraction, polarization, etc., which are basically the results of interaction of radiation with another radiation. In the aforesaid phenomena, the radiations behave as waves. Again in case of some other phenomena, e.g., black-body radiation, photoelectric effect, Compton effect, etc., the radiations behave as a set of corpuscles (or particles), i.e., discrete entities called photons or light quanta.

These phenomena are the results of interactions of matter with energy (or radiation). So, one can observe that radiations can act as both particles and waves at different instants of time but not simultaneously. As nature exhibits examples of symmetry in its behaviour which is called the principle of symmetry, one can accordingly expect that particles also should exhit dual nature like radiations. That the matter also should possess the properties to behave as particle and wave under suitable conditions.

The wave particle duality of matter was first predicted by the French theoretical physicist Pierre Louis de Broglie in the year of 1924. This theoretical concept of de Broglie was later experimentally verified by

C.J. Davison and L.H. Germer of USA in 1927 and also by G.P. Thomson of UK in the same year. The wave properties of matter can be reconciled with the particle properties of it by superposition of waves of various wavelengths to form a group of waves known as wave packet. **A wave packet is able to represent a particle in motion since any moving particle remains confined to a small region in space at any instant of time. The manifestation of wave-particle duality of radiations becomes transparent to any one if one considers the entire e.m. spectrum.** The radio waves (wavelength λ ~ a few hundred metres) are there at lower frequency region and spreads over a large volume in the space (Fig. 9.9].

Fig. 9.9 Electromagnetic spectrum-wave nature is prominent in the low frequency region whereas particle nature is prominent in the high frequency region of the spectrum.

So the available energy at any point is insignificantly small and for this reason, the particle nature cannot be observed. This is because the particles remain confined in a very small area unlike the radio waves. Again at higher frequency end of the e.m. spectrum there are ultraviolet, X-rays and γ-rays (wavelength λ ~ a few angstroms) where the waves are located in a very small region of space due to their very short wavelength and hence the energy at a particular point assumes a large value. As a result of this fact, the wave property is less noticeable as compared to the particle properly. Thus, at low frequency (or large wavelength) region the wave, nature is prominent whereas at high frequency (or small wavelength) region the particle nature is dominant. The visible region is the transition region where both, the wave and particle aspects are observed depending on some suitable conditions.

9.5.1 Bohr's Quantum Condition and Broglie Hypothesis

As we have just visualized the em spectrum from higher wavelengths to lower wavelengths, one can similarly visualize the matter starting from extremely large celestial bodies to extremely small subatomic particles. In this case the particle nature is prominent for big celestial bodies whereas the wave nature becomes prominent in case of subatomic particles like electrons. Louis de Broglie in 1925 proposed the dual character for electrons having been guided by the aforesaid logic. He was also trying to discover the underlying significance of Bohr's quantum condition for atomic structure. His idea was to fit an exact number of standing waves (or half waves) into the Bohr orbit in analogy with the integral number of waves (or half-waves) in a string stretched between two rigidly fixed ends. (Fig. 9.10)

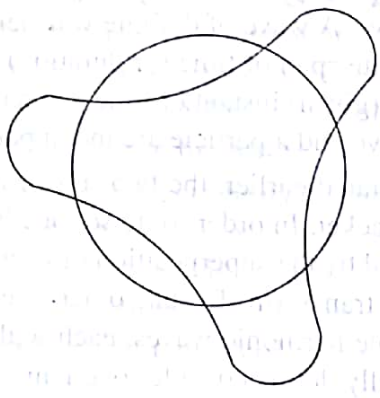


Fig. 9.10 An allowed orbit of an atom accommodates integral number of electron waves.

Thus if r_n is the radius of the nth Bohr orbit, then one has to write $2\pi r_n = n\lambda$ where n is an integer in order that an integral number of waves (with wavelength λ) may be fitted into the orbit. Comparing this with Bohr's quantum condition (i.e., $mvr_n = n\hbar$), one can obtain

$$mv\left(\frac{n\lambda}{2\pi}\right) = n\hbar$$

or, $$\lambda = \frac{2\pi\hbar}{mv}$$

or, $$\lambda = \frac{h}{mv} = \frac{h}{p} \quad ...(9.39)$$

where $p = mv$ is the momentum of the electron. $\left[\text{Bohr's quantum condition (or postulate) states that the angular momentum of an orbiting electron of an atom is equal to the integral multiple of } \hbar \text{ where } \hbar = \frac{h}{2\pi}\right]$

Thus, de Broglie put forward a proposal known as de Broglie's hypothesis which states that a material particle with energy E and momentum p may exhibit characteristics of a wave of wavelength λ. And this wavelength λ is called de Broglie wavelength. Frequency ν of the particle is determined by Planck's formula

$$E = h\nu \quad ...(9.40)$$

Equation (9.39) may be written as

$$p = \hbar k \quad ...(9.41)$$

$$\left[\because p = \frac{h}{\lambda} = \frac{h}{2\pi} \times \frac{2\pi}{\lambda} = \hbar k\right]$$

And Eq. (9.40) may be written as

$$E = \hbar\omega \quad ..(9.42)$$

$$\left[\because E = h\nu = \frac{h}{2\pi} \times (2\pi\nu) = \hbar\omega\right]$$

Here, $k = \frac{2\pi}{\lambda}$ and $\omega = 2\pi\nu$. **k is known as wave number or propagation vector and ω is called circular frequency**. The wave and particle characters of an entity (such as an electron or a photon) are mutually exclusive. A wave of definite wavelength (and frequency) classically is of infinite extent in space and also is of infinite span of time (or duration). On the other hand, a particle is localized at a definite point in space at a fixed (given) instant of time and has a definite momentum p and energy E. Hence, the basic characteristics of a wave and a particle are incompatible.

As stated earlier, the two sets of incompatible characters can have reconciliation through the concept of wave packet. In order to do so, one has to localize the wave in a finite region in space. And this goal can be achieved by the superposition of waves of various wavelengths with each other by the well-known method of Fourier transform. To state otherwise, one complex real wave can be expressed as superposition of a number of simple harmonic waves, each with slightly different velocity and wavelength. In order to grasp this idea pictorially, let us consider one pair of sinusoidal waves with slightly different velocity and wavelength and superpose them as shown in the Fig. 9.11.

As can be seen in the figure, following superposition of two waves a resultant wave (curve c) is produced which shows regularly recurring humps which repeat themselves. And due to this event, the energy in the space gets redistributed and confined in the humps. By increasing the number of participating waves one can show that the energy practically can be confined in a single hump. **Such a hump is termed as wave pocket.** And the wave pocket then represents the corresponding particle in space.

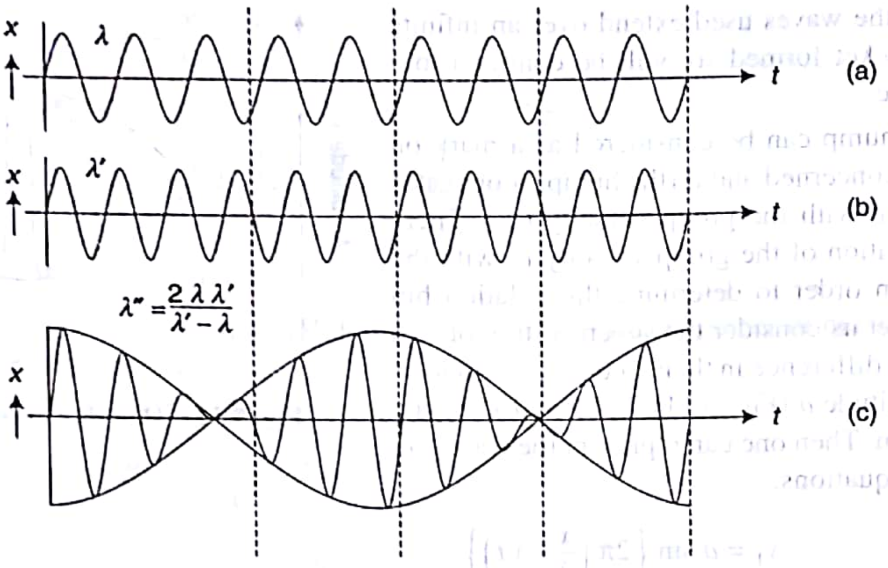


Fig. 9.11 Superposition of two sinusoidal waves of slightly different frequencies [(a) and (b)]. Curve (c) represents the resultant motion. $x = a \sin(\omega t)$.

9.5.2 Phase and Group Velocities

The phase velocity of a monochromatic wave is that velocity with which any crest or any trough of the wave propagates in a given medium. Let us denote the phase velocity by u. In a dispersive medium, the phase velocity u is different for different wavelengths. One cannot measure the phase velocity by any experimental method. For a completely monochromatic wave it is not possible to distinguish between the successive crests or troughs. They all look exactly alike. It is well-known that during the propagation of radio waves, signals are transmitted after modulation, either through the process of amplitude or that of frequency modulation. The production of amplitude-modulated signals involves the superposition of a number of plane waves with different frequencies, centering about a mean frequency. The energy carried by such modulated signals gets transmitted with a velocity called **group velocity** which is different from the phase velocity of any component wave and let us denote this velocity by v_g. The production of such modulated signals essentially constitutes affixing a distinguishing mark on the wave train. And this distinguishing mark helps one to follow the progress of the wave with time easily and to make measurements with parameters of the wave.

The simplest way of signalling a wave train is the well-known method of producing beats by superposition of two waves of slightly different wavelengths (or frequencies). When it is done, the two waves reinforce each other to produce humps at certain points in space at regular intervals of time with relatively weaker disturbances in between two humps (Fig. 9.11). As has been stated above, if the number of superposed waves increases, the regions with prominent humps will be narrower but the intervening regions of weaker disturbances will become broader. In the limit, the superposition of an infinite number of waves with continuously varying frequncies extending over a finite range produces only one hump within a very narrow region of space with practically no disturbance at any other point in the space. **One so produced hump is called a wave-packet.** This method of producing a wave packet is called the method of Fourier integral, Figure 9.12 shows one such wave-packet.

By suitably distributing the amplitudes of the various component waves, any arbitrary non-periodic wave form may be reproduced by the method of Fourier integral. It deserves to be noted here that if in this case

the frequencies of the waves used extend over an infinite range, the wave-packet formed so, will be confined to a single point in space.

The prominent hump can be considered as a mark on any wave which is concerned and it (the hump) propagates through the medium with the group velocity (v_g). There exists a definite relation of the group velocity v_g with the phase velocity u. In order to determine this relationship between v_g and u, let us consider the superposition of two waves having slight difference in their frequencies ν and ν' with the same amplitude a (Fig. 9.11). Let λ and λ' be the wavelengths of them. Then one can represent the waves by the following two equations:

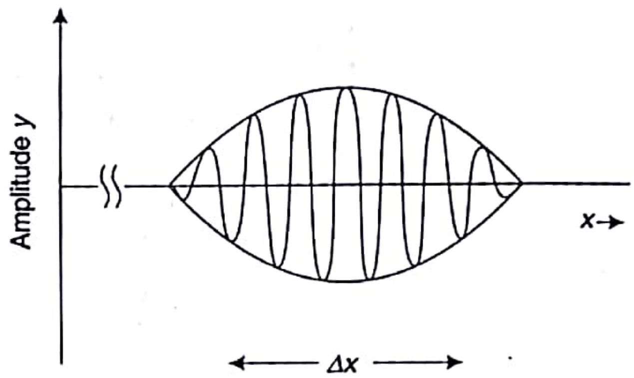


Fig. 9.12 One-dimensional wave-packet.

$$y_1 = a \sin\left\{2\pi\left(\frac{x}{\lambda} - \nu t\right)\right\}$$

or, $$y_1 = a \sin(kx - \omega t) \qquad \text{...(9.43)}$$

and $$y_2 = a \sin\left\{2\pi\left(\frac{x}{\lambda'} - \nu' t\right)\right\}$$

or, $$y_2 = a \sin(k'x - \omega' t) \qquad \text{...(9.44)}$$

where $\omega = 2\pi\nu$ and $\omega' = 2\pi\nu'$ are the circular frequencies while $k = \frac{2\pi}{\lambda}$ and $k' = \frac{2\pi}{\lambda'}$ are the wave numbers of the two waves.

The resultant wave on superposition of the two waves can be obtained as follows:

$$y = y_1 + y_2 = a \sin(kx - \omega t) + a \sin(k'x - \omega' t)$$

or, $$y = 2a \cos\left(\frac{k-k'}{2}x - \frac{\omega - \omega'}{2}t\right) \sin\left(\frac{k+k'}{2}x - \frac{\omega + \omega'}{2}t\right) \qquad \text{...(9.45)}$$

The superposed resultant wave of equation (9.45) has a wave number $\frac{k+k'}{2}$ and circular frequency $\frac{\omega + \omega'}{2}$. As can be seen from the said superposed equation, the amplitude of it is not constant. Its amplitude varies according to the cosine term which shows that it varies slowly with the circular frequency $\frac{(\omega - \omega')}{2}$. The propagation velocity of the maxima in the amplitudes is given by $\frac{(\omega - \omega')}{(k - k')}$. On the other hand, the phase velocity of propagation of the superposed wave is $\frac{(\omega + \omega')}{(k + k')}$. Since the circular frequencies ω and ω' of the participating waves are almost equal. One can write this phase velocity as

$$u = \lim_{\omega \to \omega'} \frac{(\omega + \omega')}{(k + k')} = \frac{\omega}{k} = \nu\lambda$$

or, $$u = \frac{\omega}{k} = \nu\lambda \qquad \ldots(9.46)$$

On the other hand, the group velocity v_g can be expressed as follows:

$$v_g = \lim_{\omega \to \omega'} \frac{(\omega - \omega')}{(k - k')} = \frac{d\omega}{dk}$$

or, $$v_g = \frac{d\omega}{dk} \qquad \ldots(9.47)$$

This equation (i.e., Eq. (9.47)) is also valid, in general for an infinite number of monochromatic waves of continuously varying frequencies.

Differentiating Eq. (9.46) with respect to k, one gets

$$\frac{du}{dk} = \frac{1}{k}\frac{d\omega}{dk} - \frac{\omega}{k^2}$$

or, $$\frac{du}{dk} = \frac{1}{k} \cdot v_g - \frac{1}{k} \cdot u$$

or, $$k\frac{du}{dk} = v_g - u$$

or, $$v_g = u + k\frac{du}{dk}$$

or, $$v_g = u - \lambda\frac{du}{d\lambda} \qquad \ldots(9.48)$$

since $k = \dfrac{2\pi}{\lambda}\left[\text{and } k\lambda = 2\pi \Rightarrow k\,d\lambda + \lambda\,dk = 0. \text{ Hence } \dfrac{dk}{k} = -\dfrac{d\lambda}{\lambda}\right]$

For light waves in the free space there is no dispersion. Hence $\dfrac{du}{dk} = 0$, so one gets $v_g = u = c$ where $c = 3 \times 10^{+8}\ \text{ms}^{-1}$ is the velocity of light in free space.

The group velocity v_g is equal to the phase velocity u (i.e., $v_g = u$) in any homogeneous elastic medium in general.

Group and Particle Velocities

The mathematical expression of the de Broglie hypothesis is given by

$$\lambda = \frac{h}{p} \quad \text{[vide Eqn. (9.39)]}$$

or, $$p = \frac{h}{\lambda} = \frac{h/(2\pi)}{\lambda/(2\pi)} = \frac{h}{2\pi} \times \frac{2\pi}{\lambda}$$

or, $$p = \hbar k \quad \text{[vide Eqn. (9.41)]}$$

where $k = \dfrac{2\pi}{\lambda}$, the definition of the wave number.

$\therefore$ $$k = \frac{p}{\hbar}$$

According to Einstein's mass–energy conversion relation, we get,

$$E = mc^2$$

or, $$mc^2 = E = h\nu = \frac{h}{2\pi} \times 2\pi\nu = \hbar\,\omega$$

or, $$\omega = \frac{mc^2}{\hbar}$$

Now, the phase velocity of the de Broglie wave representing the particle is given by

$$u = \frac{\omega}{k} \quad \text{[Eqn. (9.46)]}$$

or, $$u = \frac{mc^2/\hbar}{p/\hbar} = \frac{mc^2}{p} = \frac{mc^2}{mv}$$

or, $$u = \frac{c^2}{v}$$

Since the particle velocity v is less than the speed of the light in free space (c), the phase velocity u is greater than the speed of light in free space. Thus, the particle and the phase of the wave will not accompany each other. Although, the phase velocity is greater than the speed of light, it does not contradict the special theory of relativity because the phase waves do not carry energy.

The velocity of the particle is given by

$$v = \frac{p}{m} = \frac{pc^2}{mc^2} = \frac{pc^2}{E}$$

Now making use of the relations $p = \hbar k$ and $E = \hbar\,\omega$, one can get $dp = \hbar\, dk$ and $dE = \hbar\, d\omega$

Hence, one can get the rate of change of p with respect to E as follows:

$$\frac{dp}{dE} = \frac{dk}{d\omega} \quad \text{or} \quad \frac{d\omega}{dk} = \frac{dE}{dp}$$

or, $$v_g = \frac{dE}{dp} \quad \left[\because v_g = \frac{d\omega}{dk}, \text{ Eqn. (9.47)}\right]$$

The relativistic energy of a particle is given by

$$E = \sqrt{p^2 c^2 + m_o^2 c^4}$$

or, $$E = p^2 c^2 + m_o^2 c^4$$

Now differentiating both sides with respect to p one may get

$$2E\frac{dE}{dp} = 2c^2 p$$

or, $$\frac{dE}{dp} = \frac{pc^2}{E}$$

or, $$v_g = \frac{pc^2}{(mc^2)} \quad \left[\because v_g = \frac{dE}{dp}\right]$$

or, $$v_g = \frac{p}{m} \quad \text{or,} \quad v_g = \frac{mv}{m}$$

∴ $$v_g = v$$

That is, the group velocity related to a particle wave is equal to the particle velocity itself. Thus, **logically one can represent a particle by the wave-packet, i.e., the wave-packet represents the concerned moving particle.**

9.5.3 Matter Wave: de Broglie Wave

It is well known that in a real case, no wave can be extended infinitely in space. Even light waves are of finite spatial extent since the light emission process of an excited atom lasts for a time of the order of 10^{-8} seconds. During this short interval, the emitted wave train extends over a distance of approximately 300 cm. Thus, light signals propagate with the group velocity instead of the phase velocity. The energy, which is associated with the wave and carried by it, is transported with the group velocity. And hence, all the quantities, which are physically measurable and associated with the wave packet, propagate with the group velocity. On the other hand, the phase velocity is the velocity of propagation of the phase of the disturbance. In case of a light wave, the phase velocity may exceed that of light in free space, i.e., c. But the group velocity v_g always remains less than the light velocity c. And this should be the case for any physically measurable quantity because according to the special theory of relativity no real body can have a velocity which exceeds that of light in free space.

Having taken this cue from the facts stated above, de Broglie proposed that the particle velocity v should be taken equal to the group velocity v_g of the concerned wave:

$$v = v_g \qquad \ldots(9.49)$$

Now, using the relation $E = \hbar w$, the energy of the particle of rest mass m_o can be written as

$$E = mc^2 = \frac{m_o c^2}{\sqrt{1-\beta^2}} = h\nu = \hbar\omega \quad \left[\text{Here } \beta = \frac{v}{c}\right]$$

or,

$$\omega = \frac{m_o c^2}{\hbar\sqrt{1-\beta^2}} \qquad \ldots(9.50)$$

The phase velocity of the wave is given by

$$u = \nu\lambda = \frac{\omega}{k} = \frac{m_o c^2}{\hbar k\left(\sqrt{1-\beta^2}\right)} \qquad \ldots(9.51)$$

Now, differentiating with respect to k, one gets

$$\frac{du}{dk} = -\frac{m_o c^2}{\hbar k^2\sqrt{1-\beta^2}} + \frac{m_o c^2 \beta}{\hbar k(1-\beta^2)^{3/2}}\cdot\frac{d\beta}{dk} \qquad \ldots(9.52)$$

Then from equations (9.48) and (9.49) one can write

$$v_g = u + k\frac{du}{dk} = v$$

Now, with the help of equations (9.51) and (9.52), the above equation can be written as

$$v = \frac{m_o c^2}{\hbar k(1-\beta^2)^{1/2}} - \frac{m_o c^2}{\hbar k(1-\beta^2)^{1/2}} + \frac{m_o c^2\beta}{\hbar(1-\beta^2)^{3/2}}\cdot\frac{d\beta}{dk}$$

or,

$$v = \frac{m_o c^2\beta}{\hbar(1-\beta^2)^{3/2}}\cdot\frac{d\beta}{dk}$$

Hence,

$$dk = \frac{m_o c^2}{\hbar}\cdot\frac{d\beta}{(1-\beta^2)^{3/2}}$$

Now, integrating one gets

$$k = \frac{m_o c}{\hbar} \cdot \frac{\beta}{(1-\beta^2)^{1/2}} + C$$

where C is a constant of integration. If one assumes that the number $k = 0$ when particle velocity $v = 0$ (or $\beta = 0$), then $C = 0$.

So, one gets

$$k = \frac{m_o c}{\hbar} \cdot \frac{(\beta)}{\sqrt{(1-\beta^2)}} + 0$$

or,
$$k = \frac{m_o}{\sqrt{1-\beta^2}} \cdot \frac{(\beta c)}{\hbar}$$

or,
$$k = \frac{mv}{\hbar} = \frac{p}{\hbar}$$

or,
$$p = \hbar\, k = \frac{h}{2\pi} \times \frac{2\pi}{\lambda}$$

$\therefore$
$$p = \frac{h}{\lambda} \quad \text{...(9.53)}$$

As was seen earlier, the relation (9.53) between the momentum p of a particle and the wavelength λ of its de Broglie wave is consistent with Bohr's quantum condition (i.e., $mvr_n = n\hbar$). Eq. (9.53) which is the same as Eq. (9.39) is the famous de Broglie equation. This gives the mathematical relationship between the momentum of the particle p (which is a quantity characteristic of the particle nature) and the wavelength λ (which is a quantity characteristic of the wave nature).

9.5.4 Relation between Phase and Group Velocities

The phase velocity of de Broglie waves is given by

$$u = \nu\lambda = \frac{\omega}{k} \quad \text{...(9.54)}$$

Now by using ideas of special theory of relatively, electronic energy and momentum are given by

$$E = mc^2 = \frac{m_o c^2}{\sqrt{1-\dfrac{v^2}{c^2}}} = \hbar\omega \quad \text{...(9.55)}$$

and
$$p = mv = \frac{m_o v}{\sqrt{1-\dfrac{v^2}{c^2}}} = \hbar k \quad \text{...(9.56)}$$

Now dividing Eq. (9.55) by Eq. (9.56) one can have

$$\frac{E}{p} = \frac{c^2}{v} = \frac{\omega}{k} \quad \text{...(9.57)}$$

Hence, from Eq. (9.46), one gets $u = \dfrac{\omega}{k} = \dfrac{c^2}{v}$

where v is the velocity of the electrons and c is the velocity of light in the free space.

According to de Broglie hypothesis, $v = v_g$, the group velocity of de Broglie waves. Hence one has

$$u = \frac{c^2}{v_g}$$

or,
$$u v_g = c^2 \qquad \ldots(9.58)$$

Equation (9.58) gives the relationship between the phase velocity of the de Broglie waves.

9.5.5 Relation between Phase Velocity *u* and de Broglie Wavelength λ

For a de Broglie wave, the phase velocity of a particle of mass m and momentum p is given by

$$u = \frac{E}{p} \quad \left[\because u = \frac{\omega}{k} = \frac{\hbar\,\omega}{\hbar\,k} = \frac{E}{p}\right]$$

The total energy of the particle in relativistic form is given by

$$E = \sqrt{p^2 c^2 + m_o^2 c^4}$$

where m_o is the rest mass of the particle.

$$\therefore \quad u = \frac{\sqrt{p^2 c^2 + m_o^2 c^4}}{p} = c\sqrt{\frac{p^2 + m_o^2 c^2}{p^2}}$$

or,
$$u = c\sqrt{1 + \frac{m_o^2 c^2}{p^2}}$$

or,
$$u = c\sqrt{1 + \left(\frac{m_o^2 c^2}{h^2}\right)\lambda^2} \qquad \ldots(9.59)$$

where $\lambda = \frac{h}{p}$

Equation (9.59) shows that $u > c$ and in free space, this is consistent with the relation $v_g = \frac{c^2}{u}$ since $v_g = v$, the velocity of the concerned particle v is always less than c. In addition to this, u is a function of the de Broglie wavelength λ even in the free space or vacuum. But for a light wave, u does not depend on wavelength λ. Because the rest mass of light photon is zero and hence for light $u = c$. This is the difference between de Broglie wave and light wave.

9.5.6 Calculation of de Broglie Wavelength of a Particle of Mass *m* and Kinetic Energy E_k

Let E_k be the kinetic energy of a particle of rest mass m_o, then its total energy is given by,

$$E = E_k + m_o c^2$$

or,
$$\sqrt{p^2 c^2 + m_o^2 c^4} = E_k + m_o c^2$$

Squaring both sides of the equation one gets,

$$p^2 c^2 + m_o^2 c^4 = E_k^2 + m_o^2 c^4 + 2m_o E_k c^2$$

or,
$$p^2 c^2 = E_k^2 + 2m_o E_k c^2$$

or, $$p = \sqrt{E_k (E_k + 2m_o c^2)} \Big/ c$$

∴ de Broglie wavelength,

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{E_k (E_k + 2m_o c^2)}}$$

or, $$\lambda = \frac{h}{\sqrt{2m_o E_k \left(1 + \dfrac{E_k}{2m_o c^2}\right)}} \qquad \text{...(9.60)}$$

So, the de Broglie wavelength λ can be calculated for any particle (e.g., electron, proton, neutron, etc.) if its kinetic energy E_k and rest mass m_o are given.

9.5.7 Characteristics of de Broglie Waves

The de Broglie waves have the following characteristics:

(a) The lighter is the particle, the greater is the de Broglie wavelength, i.e., $\lambda \propto \frac{1}{m}$.

(b) The smaller the velocity of the particle, the greater the de Broglie wavelength of the particle.

(c) The de Broglie wavelength $\lambda = \infty$, when the particle velocity $v = 0$, i.e., the waves are produced due to the motion of the particle only.

(d) The velocity of the matter wave is not constant but depends on the velocity of the particle while the velocity of the electromagnetic wave is constant.

(e) Both the wave and particle behaviours of a moving body cannot be exhibited by it simultaneously in the same experiment. One can say that the waves have particle-like properties and particles have wave-like properties and the concepts are inseparably linked.

(f) The wave nature of matter is associated with the uncertainty in position of the particle.

9.5.8 Experimental Proof of de Broglie Hypothesis

There was no experimental evidence in support of de Broglie hypothesis till 1926. In the year of 1927, two US physicists, namely, CJ Davision and LH Germer first performed an experiment successfully which conclusively demonstrated the wave character of electrons. On the other hand, in the same year, one UK physicist, GP Thomson, also confirmed the wave nature of electrons having performed an independent experiment. We are giving below the details of the Davision and Germer's experiment.

Davision and Germer's Experiment

The arrangement of performing Davision and Germer experiment is shown in the Fig. 9.13.

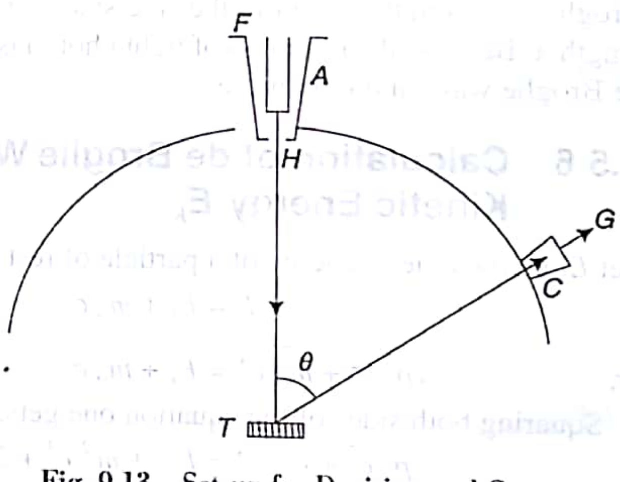


Fig. 9.13 Set-up for Davision and Germer experiment.

The beam of electrons used to perform the experiment is produced by heating a tungsten filament F. The beam is then collimated by a slit system and is accelerated by the anode A havi g small axial hole H in it. The

accelerated electron beam while emerging through this hole is incident on a single crystal of nickel T which is used as the target. The electron beam generating gun, the target crystal T and the detector are kept in a highly evacuted chamber. The electron beam hits the target normally. The scattered electrons coming out of the target T in different directions are collected by a Faraday cup C. And the intensity of the beam of scattered electrons is measured with the help of a sensitive galvanometer G. The collector (Faraday cup C) is such that all the electrons which are having the same energy in the incident beam can enter it. The ejected electrons from the nickel crystal T due to the impact of the incident electrons being of low energy cannot enter the collector (i.e., the Faraday cup C). The intensity of the scattered beam of electrons can be estimated by rotating C about an axis passing through the crystal target T.

The Results of the Experiment

The results of measurements have been shown in the Fig. 9.14. This figure consists of four parts (a), (b), (c) and (d). Each part is, in fact, a special type of plot of intensity of the scattered beam of electrons versus angle of scattering. This graphical representation is called polar graph. The line of the incident beam is taken as ordinate and a line normal to it is taken as abscissa. The radius vector ($\vec{r}$) of this graph is proportional to the intensity of the scattered beam of electrons. And the angle θ is measured from the ordinate (i.e., y-axis) instead of the abscissa (or x-axis). θ is the angle between the ordinate the radius vector $\vec{r}$.

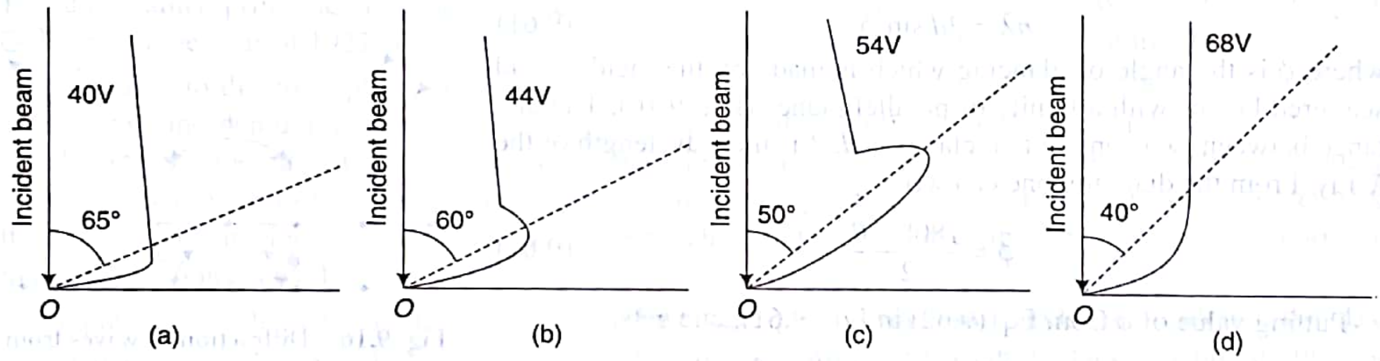


Fig. 9.14 Results of Davision and Germer experiment.

The electron beam is accelerated to various energies by applying various accelerating potentials between the filament F and the anode A. As can be seen, the graph is fairly smooth at the low values of accelerating potential but at 44 V a hump (or spur) appears on the curve at a scattering angle of $\theta = 60°$. When the accelerating potential gradually increases, the value of the radius vector at the spur also increases. The spur becomes maximum (hence maximizes the length of $\vec{r}$) at the accelerating potential of 54 V and angle of $\theta =$ 50°, when the accelerating potential increases further, the spur decreases and ultimately disappears at 68 V at a scattering angle of $\theta = 40°$. On further increase of the accelerating voltage, the graph again becomes fairly smooth. The plot of the intensity of scattered beam of electrons at 54 V against the angle of scattering has also been shown in a cartesian graph in Fig. 9.15(a) and similar plot of intensity of the beam of scattered electrons versus kinetic energy of the electrons (in eV) at angle of scattering $\theta = 50°$ has been shown in Fig. 9.15(b).

Observations of the Experiment

From both the Figs. 9.14 and 9.15 it is clear that a strong scattered beam is obtained for a potential difference of 54 volts at an angle of 50°.

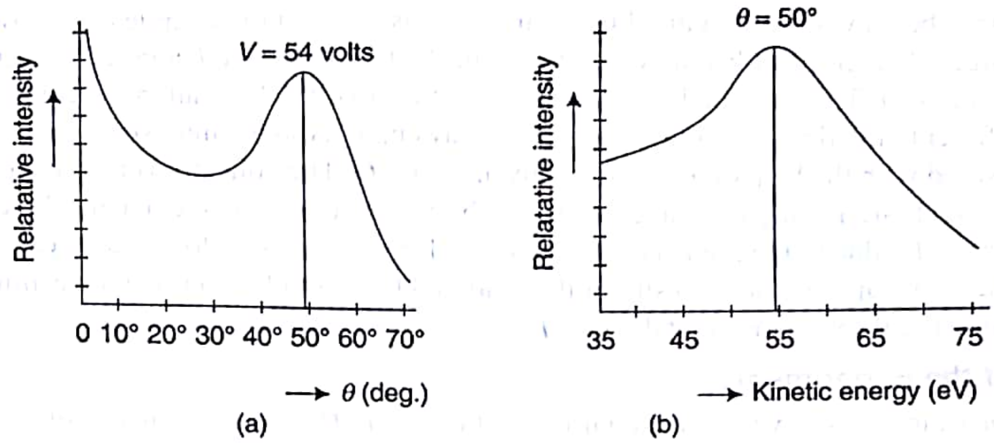


Fig. 9.15 (a) Plot of relative intensity versus angle of scattering θ at $V = 54$ volt. (b) Plot of relative intensity versus kinetic energy of electrons at $\theta = 50°$.

By the time, when Davission and Germer's experiment was performed it was very well-known that X-ray beam is diffracted from the parallel atomic planes of a crystal. And the scattered rays, in case of such diffraction, take part in constructive interference by obeying Bragg's law which is given by

$$n\lambda = 2d \sin \phi \quad \text{...(9.61)}$$

where ϕ is the angle of glancing which is made by the incident and scattered beams with a family of parallel planes (Fig. 9.16). The distance between two consecutive planes is d, λ is the wavelength of the X-ray. From the diagram, one can write,

$$\phi = \frac{180° - \theta}{2} \quad \text{...(9.62)}$$

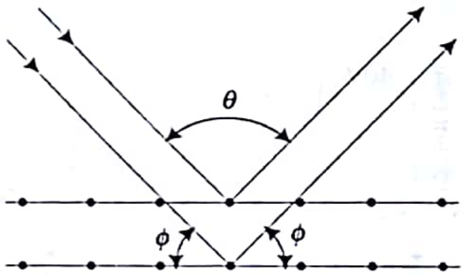


Fig. 9.16 Diffraction of waves from planes of a crystal.

Putting value of ϕ from Eq. (9.62) in Eq. (9.61), one gets,

$$n\lambda = 2d \sin \left(\frac{180° - \theta}{2}\right) \quad \text{...(9.63)}$$

Now, the experimental results of Figs 9.14 and 9.15, can be explained if one considers them to have arisen due to the constructive interference of waves scattered by the periodic arrangement of atoms in the crystal. It is possible to infer like this if one assumes that electrons are able to exhibit wave behaviour. It is to be noted that this interference takes place among the various parts of the wave associated with a single electron, scattered from different regions of the crystal. One can make the intensity of the electron beam so small that only one electron passes through the apparatus at a time. And under such condition, the pattern of the scattered electron remains the same. It very clearly establishes the fact that various parts of the de Broglie wave associated with a single electron interfere with each other. The equation (9.63) can be used to find the wavelength of the de Broglie wave linked with an electron. The interplanar distance (d) of the crystal in this case is given by $d = 0.91$ Å $= 0.091$ nm. The constructive interference takes place at $\theta = 50°$ [$\because$ one gets maximum intensity of the scattered ray at this angle]. Taking $n = 1$, one now can get

$$\lambda = 2 \times 0.091 \times \sin \left(\frac{180° - 50°}{2}\right) \text{ nm}$$

or,

$$\lambda = 1.65 \times 10^{-1} \text{ nm}$$

This is the theoretical value of the de Broglie wavelength for one electron. If p be the momentum of an incident electron, then using its relation with the kinetic energy E_k, one can write,

$$p = \sqrt{2mE_k} = \sqrt{2meV}$$

By using this expression, the de Broglie wavelength is obtained as,

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2meV}} \quad \text{...(9.64)}$$

Now, putting the values of all the parameters in the right side of Eq. (9.64) one gets

$$\lambda = 1.67 \times 10^{-10}\ \text{m} = 1.67 \times 10^{-1}\ \text{nm}$$

where $V = 54$ volts, $m = 9.11 \times 10^{-3}$ kg, $e = 1.6 \times 10^{-19}$ coul and $h = 6.626 \times 10^{-34}$ Joule-s.

This is the experimental value of the de Broglie wavelength. This experimental value of de Broglie wavelength ($\lambda = 1.67$ Å) is very close to the theoretical value of the same ($\lambda = 1.65$ Å). This excellent agreement between the theoretical and experimental values of de Broglie wavelength confirms the correctness of his hypothesis.

9.6 THE UNCERTAINTY PRINCIPLE OF HEISENBERG

The uncertainty principle (i.e., the principle of indeterminacy) was proposed by Werner Heisenberg of Germany in the year of 1927. This wonderful principle is a direct impact of the dual nature of matter.

According to the concepts of classical mechanics, a moving particle posseses a definite position as well as a definite momentum in space. Both of these two physical quantities of a particle can be definitely measured. But with the advent of the concepts of quantum mechanics in the first quarter of twentieth century, the classical view has been found to be a special case of the quantum view and it is inadequate to describe the dynamical behaviour of the microscopic objects. Though it can still describe the dynamical behaviour of macroscopic objects well.

In quantum mechanics, a particle is described by a wave-packet which surrounds the position of the classical particle and moves with a group velocity. According to Max Born's interpretation of probability, the particle may be found anywhere within the wave-packet. This means that the exact position of the particle is not certain within the limit (range) of the wave packet. And hence, one may say that the particle may be found to exist anywhere within the range of the wave-packet. Again, as we know, a wave-packet is formed by the superposition of a large number of waves having different wavelengths. The wavelength of the wave-packet cannot be said definitely. What one can say is that the wavelength of the wave-packet lies within a certain range between λ and $\lambda + \Delta\lambda$. As the momentum of a particle is related to its de Broglie wavelength given by

$$p = \frac{h}{\lambda} \quad \text{or} \quad \Delta p = -\frac{h}{\lambda^2}\Delta\lambda$$

The momentum of the particle lies in the range between p and $p + \Delta p$. So, one cannot definitely predict what is the value of the momentum of the particle. Hence, because of the wave-nature of particle there exists an uncertainty ($\Delta\bar{r}$) in position of it and uncertainty in the corresponding momentum of it ($\Delta\bar{p}$).

If the number of superposed waves as well as their range of wavelength ($\Delta\lambda$) be increased, then the spread (Δs) of the wave-packet decreases. But simultaneously the uncertainty (Δp) in momentum increases as $\Delta p \propto \Delta\lambda$. When $\Delta\lambda = \infty$, $\Delta p = \infty$ and $\Delta s = 0$, i.e., the wave-packet, in such a case, reduces to a point. In this position, there does not exist any uncertainty in the position of the particle.

Again, for a definite λ, $\Delta\lambda = 0$, this implies $\Delta p = 0$ ($\because \Delta p \propto \Delta\lambda$). In such a situation, there is no uncertainty in the momentum of the particle, but in this case $\Delta s = \infty$, i.e., the uncertainty in position is infinite since the associated wave with the particle may now extend upto infinity.

9.6.1 The Statement of the Uncertainty Principle

The uncertainty principle can be stated as follows:

It is fundamentally impossible to determine simultaneously the position as well as momentum of a particle to an accuracy which is better than one quantum of action h.

The mathematical expression for the uncertainty principle can be written as

$$\Delta s \Delta p \geq \hbar \qquad \text{...(9.65)}$$

where $\hbar = h/2\pi = 1.054 \times 10^{-31}$ Js, h being the Planck's constant ($h = 6.62 \times 10^{-34}$ Js). Here Δs is the error (i.e., uncertainty) in the determination of position of the particle and Δp is the error (i.e., uncertainty) in the determination of momentum. If the position of the particle is determined accurately, then Δs becomes smaller and consequently Δp becomes larger and vice versa. It is to be noted here that the uncertainties do not exist in apparatus used for measurement of position or momentum but they exist in the nature itself.

We have so far considered the uncertainty in one-dimensional space but it can be extended in three-dimensions also which is given by the following three equations:

$$\left.\begin{aligned} \Delta x\, \Delta p_x &\geq \hbar \\ \Delta y\, \Delta p_y &\geq \hbar \\ \Delta z\, \Delta p_z &\geq \hbar \end{aligned}\right\} \qquad \text{...(9.66)}$$

It should be noted that the uncertainty relations do not exist for any arbitrary pair of position and momentum. The uncertainty relation exists for the following pairs:

$$(x, p_x), (y, p_y) \text{ and } (z, p_z)$$

but it does not exist for the following pairs:

$$(x, p_y), (x, p_z), (y, p_z), (y, p_x), (z, p_x), (z, p_y)$$

The position and momentum pairs for which uncertainty relation exists are known as canonically conjugate pairs.

9.6.2 The Uncertainty Relation for Time and Energy

The time–energy uncertainty principle can be stated as follows:

In any simultaneous measurement of time and energy of a moving particle, the product of the uncertainty in the two physical quantities is equal to $\hbar$ ($= h/2\pi$) or greater than it.

i.e.,
$$\Delta E\, \Delta t \geq \hbar \qquad \text{...(9.67)}$$

where ΔE is the uncertainty in the measurement of energy and Δt is the uncertainty in the measurement of time.

9.6.3 The Exact Uncertainty Product

So far, we have observed that the product of uncertainties of a conjugate pair of variables is of the order of or greater than $\hbar$ (i.e., Planck's constant divided by 2π). But rigorous mathematical calculations show that the minimum value of the product of uncertainties in a conjugate pair of variables is actually $\hbar/2$. So, one can write

$$\Delta s\, \Delta p \geq \hbar/2 \qquad \text{...(9.68)}$$

And when the above one-dimensional case is extended to three-dimensions, one gets

$$\left.\begin{aligned}\Delta x\,\Delta p_x &\geq \hbar/2\\ \Delta y\,\Delta p_y &\geq \hbar/2\\ \Delta z\,\Delta p_z &\geq \hbar/2\end{aligned}\right\} \quad ...(9.69)$$

And for the conjugate pair of time and energy, the exact relation becomes

$$\Delta E\,\Delta t \geq \hbar/2 \quad ...(9.70)$$

The mathematical proof of these relations (9.69) and (9.70) is beyond the scope of this book.

9.6.4 The Physical Significance of the Uncertainty Principle

The position–momentum uncertainty relation $\left(\Delta x\,\Delta p_x \geq \frac{\hbar}{2}\right)$ leads us to draw the following inferences:

(a) If one can measure the position of a particle accurately (i.e., $\Delta x = 0$), then the uncertainty in the measurement of its momentum becomes infinity (i.e., $\Delta p_x = \infty$) at the same moment of time.

(b) If one can measure the momentum of a particle accurately (i.e., $\Delta p_x = 0$), then the uncertainty in the measurement of its position becomes infinity (i.e., $\Delta x = \infty$) at the same moment of time.

(c) For a particle of mass m moving with a velocity v, the position momentum uncertainty relation becomes

$$(\Delta x)\,(m\,\Delta v_x) \geq \frac{\hbar}{2}$$

or,
$$\Delta x\,\Delta v_x \geq \frac{\hbar}{2m} \quad ...(9.71)$$

Hence, for a heavily massive body,

$$\frac{h}{m} \approx 0.$$

So, from the relation (9.70), one can write

$$\Delta x\,\Delta v_x = 0$$

For such a particle both position and momentum can be measured accurately. This is true for macroscopic bodies whose motions are the matter of discussion in classical mechanics. Thus the uncertainty relation does not play any role in classical mechanics. On the other hand, for the microscopic objects, such as electron, proton, neutron, atom etc., the quantum mechanics is applicable. In this case, the uncertainty relation is a reality in describing the dynamical behaviour of the microscopic objects.

(d) From the relation (9.70), one can write

$$\Delta t = \frac{\hbar}{2\Delta E}$$

If the uncertainty in the measurement of energy of a system (ΔE) becomes maximum in a particular state, then from the above relation, Δt becomes minimum, i.e., the system remains for a minimum time in the state.

Again,
$$\Delta E = \frac{\hbar}{2\Delta t}$$

The above relation shows that if a system remains in a particular state for a maximum interval of time, then the uncertainty in the measurement of the energy of the system becomes minimum.

9.6.5 Applications of the Uncertainty Principle

(a) Calculation of the binding energy of an electron in an atom by using the uncertainty principles

When an electron revolves round the nucleus of an atom in an orbit of radius r, the uncertainty in the position of the electron Δx is given by

$$\Delta x = r \quad [\because \text{the position is measured from the centre of the nucleus}]$$

Now, one can write the position momentum uncertainty relation as

$$\Delta x \, \Delta p_x \geq \frac{\hbar}{2}$$

or,
$$\Delta p_x \geq \frac{\hbar}{2\Delta x}$$

or,
$$\Delta p_x \geq \frac{\hbar}{(2r)} \quad [\because \Delta x = r]$$

Hence, the minimum value of the momentum of the particle is

$$p = \Delta p_x = \frac{\hbar}{2r}$$

Now, taking $r \approx 10^{-10}$ m.

$$p = \frac{h}{4\pi \times 10^{-10}} \approx 0.527 \times 10^{-24} \text{ kg ms}^{-1}$$

The kinetic energy corresponding to this non-relativistic momentum for an electron is given by

$$E_k = \frac{p^2}{2m_o} = \frac{(0.527 \times 10^{-24})^2}{2 \times 9.1 \times 10^{-31}}$$

or,
$$E_k = 1.5 \times 10^{-19} \text{ J} \approx 1 \text{ eV}$$

Again, the potential energy of an electron in the field of the nucleus with atomic number Z is given by

$$V = -\frac{Ze^2}{4\pi E_o r}$$

or,
$$V = -\frac{Z(1.6 \times 10^{-19})^2}{4 \times 3.14 \times 8.85 \times 10^{-12} \times 10^{-10}} \text{ J}$$

or,
$$V \approx -14.4\, Z \text{ eV}$$

So, the total energy of the electron in its orbit is given by

$$E = E_k + V = (1 - 14.4\, Z) \text{ eV}$$

i.e.,
$$E = (1 - 14.4\, Z) \text{ eV}$$

So, for hydrogen atom, $[\because Z = 1]$

$$E = 1 - 14.42 = -13.4 \text{ eV}$$

And for helium atom $Z = 2$, $E = (1 - 28.8)$ eV $= - 27.8$ eV. These two binding energy values calculated from uncertainty principle agree with the binding energy of the outer most electron (s) in hydrogen atom (– 13.6 eV) and helium atom (– 24.6 eV) respectively.

(b) No electron (with separate entity) exists in the nucleus of an atom

The radius (r) of the nucleus of an atom is in the order of 10^{-14} m.

So, if one electron (with separate entity) exists inside the nucleus of the atom, then the uncertainty in the position of this electron is given by

$$\Delta s = \text{diameter of the nucleus}$$

or, $$\Delta s \approx 2r \approx 2 \times 10^{-14} \text{ m}$$

From, the principle of uncertainty, the uncertainty in momentum of the said electron is given by

$$\Delta p \geq \frac{\hbar}{\Delta s}$$

or, $$\Delta p \geq \frac{h}{2\pi \Delta s}$$

or, $$\Delta p \geq \frac{6.63 \times 10^{-34}}{2 \times 3.14 \times (2 \times 10^{-14})} \text{ kg ms}^{-1}$$

or, $$\Delta p \geq 5.27 \times 10^{-21} \text{ kg ms}^{-1}$$

It means that if the elementary particle (i.e., electron) has to be inside the nucleus of the atom, the minimum momentum of it must be given by

$$p_{\min} = 5.27 \times 10^{-21} \text{ kg ms}^{-1}$$

The minimum energy of an electron (with separate entity) of mass m is obtained from the relativistic formula (of energy) as

$$E = \sqrt{p^2 c^2 + m_o^2 c^4}$$

or, $$E = \sqrt{(5.27 \times 10^{-21})^2 \times (3 \times 10^{+8})^2 + (9.1 \times 10^{-31})^2 \times (3 \times 10^8)^4} \text{ kg ms}^{-1}$$

or, $$E \cong 10 \text{ MeV}$$

So, if one electron (with separate entity) remains in the nucleus of the atom, its energy must be of the order of 10 MeV. But it is known from the experimental data that the electrons get emitted from radioactive nuclei during β-decay with energy of about 4 MeV.

Therefore, electrons cannot exist within the nucleus of an atom rather they get generated instantaneously just before their emission.

(c) Protons and neutrons can exist in the nucleus of an atom with their separate entity

If one nucleon (proton or neutron) has to be inside the nucleus of an atom its minimum momentum must be

$$p_{\min} = 5.27 \times 10^{-21} \text{ kg ms}^{-1}$$

[It can be calculated as was done in case of one electron in (a)].

For the nucleon, rest mass

$$m_o \cong 1.67 \times 10^{-27} \text{ kg}$$

So, the corresponding value of energy (kinectic energy) is given by

$$E_k = \frac{p^2}{2m_o} = \frac{(5.27 \times 10^{-21})^2}{2 \times 1.67 \times 10^{-27}} \text{ J}$$

or, $$E_k = 52 \text{ K eV}$$

As this energy E_k is smaller than the energy carried by a nucleon emitted from a nucleus, a nucleon can exist inside the nucleus.

(d) Proof of the finite width of the spectral lines by using the uncertainty principle

The Heisenberg's energy-time uncertainty relation gives us

$$\Delta E \, \Delta t \geq \hbar/2$$

Since, the lifetime of an electron in an excited state is not infinite and it is in the order of 10^{-8} s,

$\therefore$ $$\Delta t = 10^{-8} \text{ s}$$

Hence $$\Delta E \geq \frac{\hbar}{2\Delta t}$$

or, $$\Delta E \geq \frac{6.63 \times 10^{-34}}{2 \times 2 \times 3.14 \times 10^{-8}} \text{ J}$$

or, $$\Delta E \geq 0.53 \times 10^{-27} \text{ J}$$

This implies that the excited (energy) levels of an atom given by $\Delta E = \dfrac{\hbar}{2\Delta t}$ must have a finite width (i.e., finite energy spread) instead of a sharp line.

(e) Calculation of the uncertainty in frequency of a radiation emitted by an atom

If the life-time of an excited state of an atom be Δt and ΔE be the uncertainty in energy of the state, then according to the uncertainty relation, one gets,

$$\Delta E \, \Delta t \geq \frac{\hbar}{2} \quad \text{or} \quad \Delta E \geq \frac{\hbar}{2\Delta t}$$

or, $$\Delta E \geq \frac{h}{4\pi\Delta t}$$

But the energy $E = h\nu$. So, $\Delta E = h\Delta\nu$, where $\Delta\nu$ is the uncertainty in the frequency of the emitted photon.

$\therefore$ $$\Delta\nu = \frac{\Delta E}{h} = \frac{h}{4\pi\Delta t} \cdot \frac{1}{h} = \frac{1}{4\pi\Delta t}$$

Now, the average life-time of the excited state of an atom is given by

$$\Delta t = 10^{-8} \text{ s.}$$

$\therefore$ $$\Delta\nu = \frac{1}{4\pi \times 10^{-8}} = 0.8 \times 10^{7} \text{ Hz.}$$

Thus, the radiation emitted by an atom cannot have a definite frequency (ν), instead it will have a variable frequency lying within the range of frequency between $\nu - \Delta\nu$ and $\nu + \Delta\nu$. In other words, the radiation emitted will have a band of frequency.

9.7 THE OLD QUANTUM THEORY

Classical mechanics not only failed to explain the phenomena like black-body radiation, photoelectric effect and Compton effect but also failed to explain the phenomenon of stability of atoms. According to Rutherford, an atom consists of a positively charged heavy nucleus surrounded by a set of negatively charged electrons which revolve around the nucleus in circular orbits. These electrons experience strong attraction towards the positively charged heavy nucleus. As a result of the action of this strong attractive force, they should come closer to the nucleus with time. Secondly, the energy of these negatively charged revolving electrons should decrease continuously since an accelerating charged particle always radiates energy in the form of e.m. radiations. And due to this continuous loss of energy through radiation, the orbital electrons should come closer and closer to the nucleus until they collide with the nucleus and the atom collapses. This is an indication of instability of the atoms. But it is contrary to the fact of stability of the atoms. So, the classical mechanics fails to explain the stability of the atoms.

9.7.1 Bohr's Theory of the Hydrogen Atom

The hydrogen atom is the simplest of all atoms. It consists of a proton as its nucleus and a single electron orbiting around the nucleus. The nucleic proton is very heavy as compared to the orbiting electron and practically remains at rest. The force of attraction between the two, being electrostatic, obeys the inverse square law and the possible orbits are circular or elliptical. If the electron orbits in a circular orbit of radius r with a constant speed v, then the force of attraction is given by $\frac{1}{4\pi\varepsilon_o}\cdot\frac{e^2}{r^2}$ and it is balanced by the centrifugal force $\frac{mv^2}{r}$, where e is the electronic charge.

This fact gives us the relation,

$$mv^2 = \frac{1}{4\pi\epsilon_o}\cdot\frac{e^2}{r} \qquad \ldots(9.72)$$

which can be used to eliminate v from the expression of the energy E:

$$E = E_k + E_p$$

where E_k is the kinetic energy, E_p is the potential energy and E is the total energy of the electron.

So,
$$E = \frac{1}{2}mv^2 + \left(-\frac{1}{4\pi\epsilon_o}\cdot\frac{e^2}{r}\right)$$

[The minus sign here indicates that the potential energy is of attractive nature]

or,
$$E = \frac{1}{2}\cdot\frac{1}{4\pi\epsilon_o}\cdot\frac{e^2}{r} - \frac{1}{4\pi\epsilon_o}\cdot\frac{e^2}{r}$$

or,
$$E = -\frac{1}{2}\,\frac{1}{4\pi\epsilon_o}\cdot\frac{e^2}{r} \qquad \ldots(9.73)$$

Thus, the total energy is one-half of the potential energy.

Classically, the orbital radius r can assume any positive value, and accordingly E can occupy any value from $-\infty$ to $+0$. However, according to Bohr's first postulate, the electron can occupy only any one of the

special set of orbits among all the possible ones. He proposed that these special orbits which characterize the stationary states are those in which the angular momentum L of the electron about the centre of the orbit (i.e., the nucleus of the atom) is an integral multiple of $\hbar$ (= $h/2\pi$). As the angular momentum in a circular orbit is given by the product of the linear momentum (mv) and the orbital radius (r), so the Bohr's quantum condition can be written as

$$mvr = n\hbar \qquad ...(9.74)$$

[In other form, $\oint L d\phi = nh$, where, $L = mvr$]

where n = 1, 2, 3, ..., N and the value of n identifies the stationary states. *The integer n is known as quantum number.* The radii of the permitted orbits are now obtained by eliminating v from the Eq. (9.72) and Eq. (9.74).

For the nth orbit one gets

$$r_n = \frac{E_o n^2 h^2}{\pi m e^2} \qquad ...(9.75)$$

where r_n is the radius of the nth orbit. Substituting this value of r_n in Eq. (9.73), one gets the quantized energy levels of the hydrogen atom as follows:

$$E_n = -\frac{me^4}{8E_o^2 h^2}\left(\frac{1}{n^2}\right) \qquad ...(9.76)$$

where n = 1, 2, 3, ..., N and E_n is the nth energy state (or energy level). The state of the lowest energy (i.e., the ground state) corresponds to $n = 1$. The energies of the other ('excited') states increase with n, tending to 0 as $n \to \infty$.

Having known the energy levels, one can immediately obtain the wavelengths (or frequencies) of the hydrogen spectrum when the electron jumps from the initial state n_i to the final state n_f (where $n_f < n_i$) as follows:

$$h\nu = E_i - E_f \quad \text{or} \quad \nu = \frac{E_i - E_f}{h}$$

or,

$$\frac{c}{\lambda} = \frac{me^4}{8E_o^2 h^3}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

$\left[\text{by using Eq. (9.76) and } \nu = \frac{c}{\lambda}\right]$

or,

$$\frac{1}{\lambda} = \frac{me^4}{8E_o^2 ch^3}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

or,

$$\frac{1}{\lambda} = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \qquad ...(9.77)$$

where $R = \frac{me^4}{8E_o^2 ch^3} = 1.097 \times 10^7 \text{ m}^{-1}$ and known as **Rydberg number (or constant).**

According to the classical theory, an excited hydrogen atom can radiate energy of all wavelengths continuously. But experimentally it was observed that the excited hydrogen atom emits em radiations of certain definite wavelengths due to transition of the electron from higher energy state to lower energy states corresponding to Lyman (n_f = 1), Balmer (n_f = 2), Paschen (n_f = 3), Bracket (n_f = 4) and Pfund (n_f = 5) series. These discrete set of lines are represented by Eq. (9.77). So, it can be concluded that the classical theory fails to explain the atomic spectra.

The Bohr's quantum condition of Eq. (9.74) is a special case of the general quantum condition given by

$$\oint p_r\, dq_r = n_r h; \quad r = 1, 2, 3, \ldots, N \qquad \ldots(9.78)$$

where the integration in case of each pair of conjugate variables (i.e., generalised conjugate co-ordinates) is to be taken over one period of that particular pair. The quantum number n_r takes integral values. Bohr's general postulates along with the quantum rule of Eq. (9.78) constitute what is known as the old quantum theory.

Worked-out Examples

Black-body Radiation

Example 9.1 (a) Calculate the energy in eV of a photon of wavelength 1 Å

(b) What is the momentum of this photon?

Sol. (a) If λ be the wavelength of the photon, h be the Planck's constant and c is the velocity of light then energy of the photon $E = \dfrac{hc}{\lambda}$

Here $h = 6.62 \times 10^{-34}$ J-s, $c = 3 \times 10^8$ m/s, $\lambda = 1$ Å $= 10^{-10}$ m.

$$\therefore \quad E = \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{10^{-10}} = 19.86 \times 10^{-16} \text{ J}$$

$$= \frac{19.86 \times 10^{-16}}{1.6 \times 10^{-19}} \text{ eV} = 12.412 \text{ K eV}$$

(b) Momentum of the photon $p = \dfrac{h}{\lambda}$

or,
$$p = \frac{6.62 \times 10^{-34}}{10^{-10}} = 6.62 \times 10^{-24} \text{ kg m/s}$$

Example 9.2 Calculate the average energy of Planck's oscillator for $\left(\dfrac{\hbar\,\omega}{kT}\right) = 0.01,$

Sol. The average energy of Planck's oscillator in terms of angular frequency (ω) is

$$E_a = \frac{\hbar\omega}{\exp\left(\dfrac{\hbar\omega}{kT}\right) - 1} = \frac{\hbar\omega/kT}{\exp\left(\dfrac{\hbar\omega}{kT}\right) - 1} \times kT$$

Putting the values of $\dfrac{\hbar\omega}{kT}$ given in the problem we find

$$E_a = 0.00\,45\; kT$$

Example 9.3 Calculate the number of modes in a cube of 2 cm side in the wavelength range 4995 Å and 500 S Å.

Sol. Number of modes within the volume (V) having wavelength between λ and $\lambda + d\lambda$ is given by

$$dn = \frac{8\pi V}{\lambda^4}\, d\lambda$$

Here $V = L^3 = (2 \times 10^{-2}\ \text{m})^3$

$d\lambda = 10\ \text{Å} = 10 \times 10^{-10}$ m.

$\therefore \quad dn = \dfrac{8 \times 3.14 \times (2 \times 10^{-2})^3}{(5000 \times 10^{-10})^4} \times 10 \times 10^{-10}$

or, $\quad dn = 3.21 \times 10^{11}$

Example 9.4 Calculate the number of photons emitted per second by a 100 watt sodium lamp [Assume $\lambda = 5893$ Å of Sodium lamp]

Sol. Energy of a photon,

$$E = h\nu = \frac{hc}{\lambda}$$

$$= \frac{6.63 \times 10^{-34} \times 3 \times 10^8}{5893 \times 10^{-10}} = 3.375 \times 10^{-19} \text{ joule.}$$

Now, the rate of energy emission from a 100 watt lamp is 100 J/s. Since the energy of a photon is 3.375×10^{-19} J, the number of photons emitted per second is $\dfrac{100}{3.375 \times 10^{-19}} = 2.96 \times 10^{20}$

Example 9.5 Radiation from the moon gives, λ_m of 4700 Å and 14×10^{-6} m. What conclusion can you draw from this? [$h = 6.6 \times 10^{-34}$ Js, $k = 1.37 \times 10^{-22}$ J/K, $c = 10^8$ m/s]

Sol. We know that Wien's displacement law is

$$\lambda_m T = 0.289 \times 10^{-2} \text{ mk}$$

when $\quad \lambda_m = 4700\ \text{Å} = 4700 \times 10^{-10}$ m

or, $\quad T = \dfrac{0.289 \times 10^{-2}}{4700 \times 10^{-10}} = 6166$ K

and when $\quad \lambda_m = 14 \times 10^{-6}$ m

$$T = \frac{0.289 \times 10^{-2}}{14 \times 10^{-6}} = 207 \text{ K}$$

As the sun's radiation is reflected from the moon, $T_1 = 6166$ K represents the temperature of the sun. $T = 207$ K represents the temperature of the moon.

Compton Effect

Example 9.6 An X-ray photon is found to have its wavelength doubled on being scattered through 90°. Find the wavelength and energy of the incident photon.

Sol. We know that Compton shift

$$\lambda' - \lambda\ (= \Delta\lambda) = \frac{h}{m_o c}(1 - \cos\theta)$$

Here $\theta = 90°$ and $\lambda' = 2\lambda \quad \therefore\ \Delta\lambda = 2\lambda - \lambda = \lambda$

and wavelength $\lambda = \dfrac{h}{mc} = \dfrac{6.62 \times 10^{-34}}{9.1 \times 10^{-31} \times 3 \times 10^{8}}$

$= 0.0245$ Å

The energy of the incident photon

$$E = h\nu = \frac{hc}{\lambda} = \frac{6.62 \times 10^{-34} \times 3 \times 10^{8}}{0.0245 \times 10^{-10}}$$

$= 8.1 \times 10^{-14}$ joule.

Example 9.7 (a) At which angle will the Compton shift be maximum?

(b) Calculate the Compton wavelength in Å for an electron. [WBUT 2005]

Sol. (a) We know Compton shift

$$\Delta\lambda = \frac{h}{m_o c}(1 - \cos\theta)$$

If $\theta = 180°$; then $\Delta\lambda$ will be maximum. In that case,

$$\Delta\lambda = \frac{2h}{m_o c} = 2 \times 0.0245 \text{ Å}$$

$$= 0.049 \text{ Å}$$

(b) Compton wavelength $\lambda_c = \dfrac{h}{m_o c} = \dfrac{6.62 \times 10^{-34}}{9.1 \times 10^{-31} \times 3 \times 10^{8}}$

$= 0.0245$ Å.

Example 9.8 X-rays of wavelength 10.0 pm are scattered from a target.

(a) Find the wavelength of two X-rays scattered through 45°.

(b) Find the maximum wavelength present in the scattered X-rays.

(c) Find the maximum kinetic energy of the recoil electrons.

Sol. (a) We know Compton shift

$$\lambda' - \lambda = \frac{h}{m_o c}(1 - \cos\theta)$$

or, $$\lambda' = \lambda + \frac{h}{m_o c}(1 - \cos\theta)$$

$$= 10 + (0.293)\frac{h}{m_o c} = 10.7 \text{ pm} \quad [\because \theta = 45°]$$

(b) $\lambda' - \lambda$ is maximum when $(1 - \cos\theta) = 2$ in which case

$$\lambda' = \lambda + \frac{2h}{m_o c}$$

$$= 10 + 4.9 = 14.9 \text{ pm.}$$

(c) Kinetic energy of electron $= h(\nu - \nu') = hc\left(\dfrac{1}{\lambda} - \dfrac{1}{\lambda'}\right)$

$$= \frac{6.62 \times 10^{-34} \times 3 \times 10^{8}}{10^{-12}} \left(\frac{1}{10} - \frac{1}{14.9} \right)$$

$$= 6.54 \times 10^{-15} \text{ J}$$

$$= 40.8 \text{ K eV.}$$

Example 9.9 An *X*-ray photon of frequency 3×10^{19} Hz collides with a free electron at rest and is scattered at 60°. Find the direction in which the electron will move. $\left[\text{given } \frac{h}{m_o c} = 0.024 \text{ Å}\right]$.

Sol. Here $\nu = 3 \times 10^{19}$ Hz, $\theta = 60°$, $h = 6.62 \times 10^{-34}$ Js

Now, the angle of scattering θ and the angle of recoil of the electron ϕ are given by,

$$\cot \phi = (1 + \alpha) \tan \frac{\theta}{2}$$

where $$\alpha = \frac{h\nu}{m_o c^2} = \left(\frac{h}{m_o c} \right) \frac{\nu}{c}$$

$$= 0.024 \times 10^{-10} \times \frac{3 \times 10^{19}}{3 \times 10^{8}} = 0.242$$

$\therefore$ $$\cot \phi = (1 + 0.0243) \tan 30° = 1.242 \frac{1}{\sqrt{3}} = \frac{1.242}{1.73}$$

or, $$\tan \phi = \frac{1.73}{1.242} = 1.39$$

$\therefore$ $$\phi = 54.34°$$

Example 9.10 Show that a free electron at rest cannot absorb a photon.

Sol. If p_{ph} and p_{ec} be the momentum of a photon and that of a free electron respectively, then

$$p_{ph} = p_{ec} \quad \text{or,} \quad \frac{h\nu}{c} = p_{ec}$$

If E_{ph} and E_{ec} be the energies of a photon and a free electron respectively,

then $$E_{ph} = E_{ec} \quad \text{or,} \quad h\nu = \sqrt{(p_{ec}\, c)^2 + (m_o c^2)^2}$$

or, $$\frac{h\nu}{c} = \sqrt{p_{ec}^2 + m_o^2 c^2} > p_{ec}$$

Thus the conservation law of momentum breaks down. Hence the result.

So we may conclude that Compton scattering must occur with free electron.

de Broglie Hypothesis

Example 9.11 Calculate the wavelength associated with an electron accelerated to a potential difference of 1.25 KeV.

Sol. If *E* is the kinetic energy of the electron, the de Broglie wavelength of the wave associated with the electron is

$$\lambda = \frac{h}{\sqrt{2mE}} = \frac{6.62 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 1.25 \times 10^{3} \times 1.6 \times 10^{-19}}} \quad [\because E = eV]$$

$$\therefore \quad \lambda = 3.46 \times 10^{-11} \text{ m}$$

Example 9.12 What is the de Broglie wavelength of an electron moving with velocity, $v = \frac{3c}{5}$?

Sol. The de Broglie wavelength of a particle of mass m moving with velocity v is $\lambda = \frac{h}{mv}$.

Here, $\frac{v}{c} = \frac{3}{5}$, so the electron mass undergoes relativistic variation according to the relation,

$$m = \frac{m_o}{\sqrt{1 - v^2/c^2}}, \; m_o \text{ is the rest mass of the electron.}$$

$$\therefore \quad \lambda = \frac{h}{m_o v}\sqrt{1 - v^2/c^2} = \frac{6.63 \times 10^{-34}}{9.1 \times 10^{-31} \times \frac{3}{5}c}\sqrt{1 - \frac{9}{25}}$$

$$= \frac{6.63 \times 10^{-34} \times 4}{9.1 \times 10^{-31} \times 3 \times 3 \times 10^{8}} = 0.33 \times 10^{-11} \text{ m}$$

$$\therefore \quad \lambda = 0.032 \text{ Å}$$

Example 9.13 What is the de Broglie wavelength of an electron which has been accelerated from rest through a potential difference of 100 V.

Sol. We know $\lambda = \frac{12.25}{\sqrt{V}}$ Å

Here $V = 100$ V

$$\therefore \quad \lambda = \frac{12.28}{\sqrt{100}} = 1.225 \text{ A}$$

Example 9.14 An electron and a proton have the same de Broglie wavelength. Prove that the energy of the electron is greater than that of the proton. [WBUT 2008]

Sol. For the electron de Broglie wavelength is given by

$$\lambda_e = \frac{h}{m_e v_e} \text{ where } m_e \text{ and } v_e \text{ are the mass and velocity of the electron.}$$

And for the proton we get

$$\lambda_p = \frac{h}{m_p v_p} \text{ where } m_p \text{ and } v_p \text{ are mass and velocity of the proton}$$

But $\lambda_e = \lambda_p$

or, $$\frac{h}{m_e v_e} = \frac{h}{m_p v_p} \quad \text{or,} \quad m_e v_e = m_p v_p$$

Now, the energy of the electron is given by

$$E_e = \frac{1}{2} m_e v_e^2$$

$$= \frac{1}{2} (m_e v_e)^2 \times \frac{1}{m_e}$$

$$= \frac{1}{2} (m_p v_p)^2 \times \frac{1}{m_e}$$

$$= \frac{1}{2} m_p v_p^2 \times \frac{m_p}{m_e}$$

or, $$E_e = E_p \times \frac{m_p}{m_e}$$ where E_p is the energy of the proton

or, $$\frac{E_e}{E_p} = \frac{m_p}{m_e}$$

$\therefore$ $$E_e > E_p \quad [\because m_p > m_e]$$

So, the energy of the electron is greater than that of the proton.

Example 9.15 Show that the de Broglie wavelength for a material particle of rest mass m_o and charge q accelerated from rest through a potential difference of V volts relativistically is given by

$$\lambda = \frac{h}{\sqrt{2 m_o qV \left(1 + \frac{qV}{2 m_o e^2}\right)}}$$

Sol. Since v is relativistic so $E_k \neq \frac{1}{2} mv^2$ and so, we cannot find momentum directly from E_k. Now using relativistic formula

$$E^2 = p^2c^2 + m_o^2 c^4$$

Again $$E = E_k + m_o c^2 = qV + m_o c^2$$

$\therefore$ $$p^2c^2 = E^2 - m_o^2 c^4$$

or, $$p^2c^2 = (qV + m_o c^2)^2 - m_o^2 c^4$$

$$= q^2V^2 + 2qV m_o c^2$$

$\therefore$ $$p^2 = \frac{q^2V^2}{c^2} + 2q\,Vm_o = 2m_o qV \left(1 + \frac{qV}{2\,m_o c^2}\right)$$

$\therefore$ $$p = \sqrt{2m_o\,qV \left(1 + \frac{qV}{2m_o c^2}\right)}$$

$\therefore$ de Broglie wavelength $$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2\,m_o qV \left(1 + \frac{qV}{2m_o c^2}\right)}}$$

Example 9.16 Find out the glancing angle at which electrons of 100 eV should fall on the lattice planes of a crystal to obtain a strong Bragg reflection in the first order [lattice spacing, $d = 2.15$ Å].

Sol. The de Broglie wavelength of an electron accelerated through a potential difference V is

$$\lambda = \frac{12.25}{\sqrt{V}} = \frac{12.25}{\sqrt{100}} = 1.225 \text{ Å}$$

Again for the nth order Bragg diffraction maximum,

$$2d \sin \theta = n\lambda$$

where θ is the glancing angle of incidence.

For 1st order $n = 1$

$$\therefore \quad 2d \sin \theta = \lambda \quad \text{or,} \quad \sin \theta = \frac{\lambda}{2d} = \frac{1.225}{2 \times 2.15} = 0.285$$

$$\therefore \quad \theta = 16.5°$$

Example 9.17 A crystalline material has a set of Bragg planes separated by 1.1 Å. For 2 eV neutrons what is the highest order Bragg reflection? [Given glancing angle $\theta = 22°$]

Sol. The de Broglie wavelength of the neutron is given by,

$$\lambda = \frac{h}{m_o v} = \frac{h}{\sqrt{2m_o E_k}} = \frac{hc}{\sqrt{2(m_o c^2) E_k}}$$

$$= \frac{12400 \text{ eV Å}}{\sqrt{2 \times (940 \times 10^6 v) \times (0.083 \text{ eV})}}$$

[Here $hc = 12400$ eV Å and $m_o c^2 = 940 \times 10^6$ eV]

$$= 0.993 \text{ Å}$$

For the first-order peak,

$$d = \frac{\lambda}{2 \sin \theta} = \frac{0.993 \text{ Å}}{2 \times \sin 22°} = 1.33 \text{ Å}$$

Example 9.18 Calculate the de Broglie wavelength of a baseball of mass 1 kg, moving at a speed 10 m/s. Discuss the reason why we can't observe its wave nature.

Sol. The de Broglie wavelength is given by,

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

Here $m = 1$ kg, $v = 10$ m/s, $h = 6.62 \times 10^{-34}$ Js.

So $$\lambda = \frac{6.62 \times 100^{-34}}{1 \times 10} = 6.62 \times 10^{-35} \text{ m}$$

or, $$\lambda = 6.62 \times 10^{-25} \text{ Å}$$

To observe the diffraction effect, the dimension of the apparatus through which light passes must be comparable with the wavelength. The de Broglie wavelength of the baseball is so small that we can't expect to construct an apparatus which is capable of detecting wave nature by causing the diffraction of its de Broglie wave.

Uncertainty Principle

Example 9.19 The maximum uncertainty in the position of an electron in a nucleus is 2×10^{-14} m. Find the minimum uncertainly in its momentum, given $h = 6.63 \times 10^{-34}$ Js. [WBUT 2002]

Sol. We have from uncertainly principle

$$\Delta x\, \Delta p = \frac{h}{4\pi}$$

Now if position uncertainty is maximum, then momentum uncertainly will be minimum i.e.,

$$\Delta x_{max}\, \Delta p_{min} = \frac{h}{4\pi}$$

or,
$$\Delta p_{min} = \frac{h}{4\pi \times \Delta x_{max}} = \frac{6.62 \times 10^{-34}}{4\pi \times 2 \times 10^{-14}}$$

$$= 2.6 \times 10^{-21} \text{ kg ms}^{-1}$$

Example 9.20 What is the minimum uncertainty in the energy state of an atom if an electron remains in this state for 10^{-8} s?

Sol. Since the time available for energy measurement is 10^{-8} s, the uncertainty in time Δt s. Now from the relation $\Delta E\, \Delta t = \frac{h}{4\pi}$ we get

$$\Delta E = \frac{h}{4\pi\, \Delta t} = \frac{hc}{4\pi c\, \Delta t} = \frac{12400 \text{ eV Å}}{4\pi \times 3 \times 10^{8} \times 10^{-8} \times 10^{10}} \quad [1 \text{ Å} = 10^{-10} \text{ m}]$$

$$= 0.329 \times 10^{-7} \text{ eV.}$$

So the natural width (minimum uncertainty) of the energy state = 0.329×10^{-7} eV.

Example 9.21 Assume that an electron is inside a nucleus of radius 10^{-15} m. Calculate from the uncertainty principle the minimum kinetic energy of the electron. [WBUT 2007]

Sol. Using uncertainty relation

$$\Delta p_{min} = \frac{h}{4\pi\, \Delta x_{max}}$$

Here Δx_{max} is the diameter of the nucleus = 2×10^{-15} m.

$\therefore$
$$\Delta p_{min} = \frac{6.62 \times 10^{-34}}{4 \times 3.14 \times 2 + 10^{-15}} \text{ kg ms}^{-1}$$

$$= 0.263 \times 10^{-19} \text{ kg ms}^{-1}$$

Now
$$E_{min}^2 = p_{min}^2 c^2 + m_o^2 c^4$$
$$= (0.263 \times 10^{-19}) \times (3 \times 10^8)^2 + (9.1 \times 10^{-31})^2 \times (3 \times 10^8)^4$$

or,
$$E_{min}^2 = 6.2 \times 10^{-23}$$

$\therefore$
$$E_{min} = 7.9 \times 10^{-12} \text{ joule}$$
$$= \frac{7.9 \times 10^{-12}}{1.6 \times 10^{-19}} \text{ eV} = 49 \text{ MeV}$$

Example 9.22 An electron has a speed 4×10^5 m/s within the accuracy of 0.01 per cent. Calculate the uncertainty in position of the electron. Given $h = 6.62 \times 10^{-34}$ Js.

Sol. Uncertainty in velocity $= \dfrac{0.01}{100} \times 4 \times 10^5 = 40$ m/s

$$\Delta x\, \Delta p = \frac{h}{4\pi}$$

$$\Delta x = \frac{h}{4\pi(\Delta p)}$$

$$= \frac{6.62 \times 10^{-34}}{4 \times 3.14 \times 9.1 \times 10^{-31} \times 40} \quad [\text{here } \Delta p = m\,\Delta v]$$

$$= 1.447 \times 10^{-6} \text{ m.}$$

Example 9.23 If a component of angular momentum of the electron in a hydrogen atom is known to be 2 $\hbar$ within 5% error, show that its angular position in the plane perpendicular to the component can't be specified at all.

Sol. According to uncertainty relation

$$\Delta L\, \Delta\theta \simeq \hbar$$

Here ΔL = uncertainty in angular momentum

$\Delta\theta$ = Corresponding uncertainty in angular position

$$\therefore \quad \Delta L = \frac{5 \times 2\hbar}{100}$$

So $$\Delta\theta = \frac{\hbar}{\Delta L} = \frac{\hbar \times 100}{5 \times 2\,\hbar} = 10 \text{ radians} > 2\pi$$

The angle in a plane perpendicular to the component of angular momentum can't be greater than 2π. So, the orbital angular position of the electron in the plane perpendicular to the given component of angular momentum can't be specified at all.

Example 9.24 Imagine an electron to be somewhere in the nucleus whose dimension is 10^{-14} m. What is the uncertainty in momentum? Is this consistent with the binding energy of nuclear constituents?

Sol. For uncertainty relation

$$\Delta x\, \Delta p \simeq \hbar = \frac{h}{2\pi}$$

$$\Delta p = \frac{h}{2\pi\,\Delta x} = \frac{6.62 \times 10^{-34}}{2 \times 3.14 \times 10^{-14}} = 1.054 \times 10^{-20} \text{ kg m/s}$$

The least value of momentum equal to 1.054×10^{-20} kg m/s

The corresponding kinetic energy of electron is

$$E = pc \quad [\text{Here rest mass energy of electron is very small}]$$

$$= 1.054 \times 10^{-20} \times 3 \times 10^8 \text{ J}$$

$$= 3.162 \times 10^{-12}/1.6 \times 10^{-19} \text{ eV}$$

$$= 20 \text{ MeV}$$

Experiments shows that energy of electrons in nuclear disintegration (β – decay) is very much less than 20 MeV. Hence the uncertainty principle rules out the possibility of electrons being a nuclear constituent.

Review Exercises

Part 1: Multiple Choice Questions

1. Total energy emission per unit area per unit time from the surface of a black-body at an absolute temperature T is proportional to
   (a) $T^{3/2}$ (b) T^4 (c) T (d) $T^{5/2}$
2. Quantum nature of light emerged in an attempt to explain
   (a) radioactivity (b) pair production
   (c) black-body radiation (d) interference of light
3. The absorptive power of a black-body is
   (a) 2 (b) 0 (c) 1 (d) 3
4. Black-body radiation is
   (a) an elastic wave (b) mechanical wave
   (c) electromagnetic wave (d) none of these
5. Wien's displacement law could explain the distribution of thermal radiation in the spectrum of black-body radiation in the region of
   (a) higher wavelength (b) lower wavelength
   (c) middle wavelength (d) none of these
6. All the radiation laws can be shown to be special cases of
   (a) Wien's law (b) Stefan – Boltzmann's law
   (c) Rayleigh – Jeans law (d) Planck's law
7. Rayleigh – Jeans law could explain the energy distribution in the spectrum of black-body radiation in the region of
   (a) higher wavelength (b) lower wavelength
   (c) middle wavelength (d) X-rays
8. Average energy of an oscillator of frequency v in a cavity at an absolute temperature T is,
   (a) $\frac{kT}{hv}$ (b) kT (c) $\frac{hv}{e^{hv/kT}} - 1$ (d) $\frac{hv}{e^{hv/kT}} + 1$
9. The emissive power of a black-body kept at an absolute temperature [WBUT 2006]
   (a) T^4 (b) T^3 (c) T^5 (d) T^{-1}
10. Planck's oscillator has
    (a) continuous energy (b) discrete energy
    (c) a mixture of continuous and discrete (d) none of these

11. The wavelength of which the spectral energy density of emitted radiation at temperature T from a black-body attains maximum value, is proportional to, [WBUT 2005]

(a) $\frac{1}{T}$ (b) T (c) T^4 (d) $T^{3/2}$

12. Number of oscillation modes for the electromagnetic standing waves of frequency ν and $\nu + d\nu$ or the cavity radiation is proportional to [WBUT 2008]

(a) ν (b) ν^2 (c) ν^4 (d) $\frac{h\nu}{e^{h\nu/KT}-1}$

13. The Compton wavelength is [WBUT 2004]

(a) $\frac{h}{m_o c}$ (b) $\frac{h}{m_o c^2}$ (c) $\frac{h}{m_o c^2}$ (d) $\frac{h}{m_o c}$

14. The Compton shift depends on

(a) angle of scattering (b) material of the target
(c) wavelength of the incident X-rays (d) none of these

15. The Compton shift is maximum when the scattering angle is,

(a) 45° (b) 90° (c) 180° (d) 60°

16. Compton effect is observed if the target electron is

(a) free
(b) bound to the nucleous
(c) binding energy is very small compared to the energy of the incident photon
(d) none of these

17. In unmodified Compton line, the wavelength of the scattered photon

(a) equals the wavelength of the incident photon
(b) greater than the wavelength of the incident photon
(c) less then the wavelength of the incident photon
(d) none of these

18. The kinetic energy of the recoil electron in Compton effect is desired from the

(a) lattice energy of the metarial (b) energy of the incident photon
(c) thermal energy of the free electrons (d) rest mess of the electron

19. De Broglie proposed the hypothesis of matter wave by examining the parallelism of

(a) light and sound (b) heat and motion
(c) mechanics and optics (d) electricity and magnetism

20. The de Broglie wavelength of a particle with momentum p is,

(a) $\frac{h}{p^2}$ (b) $\frac{p^2}{h^2}$ (c) $\frac{h}{p}$ (d) $\frac{p}{h^2}$

21. The de Broglie wavelength is associated with moving

(a) charged particles (b) macroscopic particles
(c) substomic particles (d) electrically neutral particles

22. A proton and an electron are accelerated by the same potential difference. Let λ_e and λ_p denote the de Broglie wavelengths of the electron and the proton respectively, then
(a) $\lambda_e = \lambda_p$ (b) $\lambda_e < \lambda_p$ (c) $\lambda_e > \lambda_p$ (d) none of these
23. A stone is dropped from the top of a building? What happens to the de Broglie wavelength of the stone as it falls? [WBUT 2008]
(a) It increases (b) It decreases
(c) Remains constant (d) de Broglie wavelength can't be defined
24. Phase velocity and group velocity are equal when the medium is
(a) dispersive (b) non-dispersive (c) isotropic (d) none of these
25. Which statement is correct?
(a) group velocity (v_g) = phase velocity (v_p)
(b) $v_g > v_p$ (c) $v_p > v_g$ (d) $v_p = \frac{1}{v_g}$
26. The de Broglie wavelength of a moving electron subjected to a potential V is
(a) $\frac{1.26}{\sqrt{V}}$ Å (b) $\frac{12.26}{\sqrt{V}}$ Å (c) $\frac{12.26}{V}$ Å (d) none of these
27. The de Broglie wavelength of a body of mass m and kinetic energy E is [WBUT 2005]
(a) $\frac{\sqrt{2mE}}{\lambda}$ (b) $\frac{2mh}{\sqrt{E}}$ (c) $\frac{h}{\sqrt{2mE}}$ (d) $\frac{\sqrt{2mh}}{E}$
28. If visible light is used to study Compton scattering then the Compton shift will be [WBUT 2007]
(a) negative (wavelength of the scattered light will be lesser)
(b) move positive than what is observed with X-ray
(c) zero
(d) positive but not detectable in the visible window
39. An α-particle is 4 times heavier than a proton. If a proton and an α-particle are moving with the same velocity, how do their de Broglie wavelengths compare? [WBUT 2006]
(a) $\lambda_p = \lambda_\alpha$ (b) $\lambda_p = 4\,\lambda_\alpha$ (c) $\lambda_p = \frac{\lambda_\alpha}{2}$ (d) $\lambda_p = \frac{\lambda_\alpha}{4}$
30. The product of group velocity (v_g) and phase velocity (v_p) of de Broglie waves is
(a) c (b) c^2 (c) less than c (d) none of these
31. If λ_r and λ_{nr} be the relativistic and non-relativistic wavelengths of the electron, then
(a) $\lambda_r > \lambda_{nr}$ (b) $\lambda_{nr} = \lambda_r$ (c) $\lambda_{nr} > \lambda_r$ (d) $\lambda_r = \frac{1}{\lambda_{nr}}$
32. Uncertainty principle is the consequence of
(a) wave nature of particle (b) wave-particle duality
(c) particle nature of wave (d) particle-particle interaction
33. Heisenberg energy-time uncertainty relation is given by
(a) $\Delta E\, \Delta t \leq \hbar$ (b) $\Delta E\, \Delta t \geq \frac{\hbar}{2}$ (c) $\Delta E\, \Delta t \geq \frac{h}{4}$ (d) none of these

34. For very heavy particle

(a) $\Delta x\, \Delta v_x = \frac{\lambda}{2}$ (b) $\Delta x\, \Delta v_x = 0$ (c) $\Delta x\, \Delta v_x = \alpha$ (d) $\Delta x\, \Delta v_x = \frac{\hbar}{2}$

35. Uncertainty principle has significance only in

(a) macroscopic world (b) microscopic world

(c) both macroscopic and microscopic world (d) none of these

36. In the measurement of a physical entity, and experiment can exhibit only

(a) particle aspect (b) wave aspect

(c) can't exhibit wave and particle aspect simultaneously (d) all of the above

37. Uncertainty principle tells that [WBUT]

(a) a particle can have only position but no momentum

(b) a particle can have only momentum but no position

(c) one can determine simultaneously the position and momentum of a particle

(d) one cannot determine simultaneously the position and momentum of a particle

Answers

1. (b)	2. (c)	3. (c)	4. (c)	5. (b)	6. (d)	7. (a)	8. (c)
9. (a)	10. (b)	11. (a)	12. (b)	13. (d)	14. (a)	15. (c)	16. (c)
17. (a)	18. (b)	19. (c)	20. (c)	21. (c)	22. (c)	23. (b)	24. (a)
25. (c)	26. (b)	27. (c)	28. (c)	29. (b)	30. (b)	31. (c)	32. (b)
33. (b)	34. (b)	35. (b)	36. (d)	37. (d)			

Short Questions with Answers

1. What is the difference between Compton effect and photoelectric effect?

In Compton effect, the incident photon collides with loosely bound electron and gives some part of its energy to the electron. The incident photon then is scattered with its reduced energy of the scatterer.

In photoelectric effect, the incident photon is completely absorbed by the bound electron and electron is emitted from metal surface with its maximum kinetic energy which is equal to the difference of energy between the incident photon and the work function of the metal.

2. Why is Compton effect not observed with visible light?

The wavelength of visible light is $\lambda = 6000$ Å

So, energy of the visible light $= \frac{hc}{\lambda} = \frac{6.62 \times 10^{-34} \times 10^{8}}{6000 \times 10^{-10} \times 1.6 \times 10^{-19}}$ eV = 0.7 eV

But the binding energy of an electron in the atom ≃ 10 eV, so when visible light falls. on a target, it cannot liberate electrons. So we cannot observe Compton effect with visible light.

3. Show that it is impossible for a photon to give all of its energy to the electron.

We know that kinetic energy of recoil electron is,

$$E_K = h\nu \frac{2\alpha \cos^2 \phi}{(1+\alpha)^2 - \alpha^2 \cos^2 \phi}. \text{ Here } \alpha = \frac{h\nu}{mc^2}$$

and ϕ is the angle of recoil electron with incident photon.

For maximum kinetic energy $\cos\phi = 1$

So maximum value of $E_K = \dfrac{2\alpha}{1+2\alpha} h\nu$

Since $\dfrac{2\alpha}{1+2\alpha} < 1$, the maximum kinetic energy of the electron is less than $h\nu$. So we can say that photon does not transfer its entire energy to the electron.

4. **Explain modified and unmodified lines in Compton effect.**

The presence of the modified line is due to the collision of *X*-ray photon with free electrons. But we know that electrons are bound to the atom. So, when the collision occurs between the *X*-ray photon and the atom as a whole, the rest mass m of electron must be replaced by mass (M) of the atom and the equation for Compton shift (equation) must be replaced by

$$\Delta\lambda = \frac{h}{Mc}(1-\cos\theta)$$

For large value of M, $\Delta\lambda$ is nearly zero this condition gives unmodified line i.e., there is no shifting in wavelength. So wavelength of the scattered photon remains same as that of the primary.

For heavier element, the intensity of unmodified line is greater than modified line and the intensity of unmodified line is smaller than modified line.

5. **Show that Compton shift is independent of the nature of the scatterers.**

The Compton shift $\Delta\lambda = \dfrac{h}{m_o c}(1-\cos\theta)$

where θ is the scattering angle.

From expression of Compton we see that $\Delta\lambda$ depends only on the rest mass of the electron (m_o) and the scattering angle θ. So we can say that Compton shift is totally independent of the nature of the scatterer.

6. **Does the principle of uncertainty apply to macroscopic as well as microscopic bodies?**

According to uncertainty principle $\Delta p\, \Delta x = \dfrac{h}{2\pi}$, so $\Delta x\, \Delta v_x \simeq \dfrac{h}{2m\pi}$. For macroscopic object, m is very large compared to h. So uncertainty is very small. Hence macroscopic object possesses definite position and momentum. But for microscopic object, m is very small, hence uncertainty principle is applicable.

7. **What is the difference between phase velocity and group velocity?**

The velocity with which an individual wave of a wave-packet travels is called phase velocity. The velocity with which the slowly varying envelope of the modulated pattern due to a group of waves travels in a medium is called group velocity.

The phase velocity may be greator than the velocity of light. But particle (or group) velocity is always less then the velocity of light.

8. **Calculate the de Broglie wavelength of a ball of mass 250 g moving at a speed of 20 m/s. What conclusion you can draw from it?**

The de Broglie wavelength of the ball is given by

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

Here mass of the ball $m = 250 \text{ g} = \frac{1}{4}$ kg, velocity of the ball $v = 20$ m/s,

Planck's constant $h = 6.62 \times 10^{-34}$ J/s

So, $$\lambda = \frac{6.62 \times 10^{-34}}{\frac{1}{4} \times 20} = 1.23 \times 1^{-34} \text{ m}$$

$$= 1.32 \times 10^{-24} \text{ Å}$$

We know that to observe diffraction effects, the dimension of the apparatus through which light passes must be comparable with the dimension of the wavelength. The de Broglie wavelength of the ball is so small that we cannot construct an apparatus which is capable of detecting its de Broglie wavelength.

9. Show that spectral lines have a finite width.

We know from Heisenberg's energy-time uncertainty principle

$$\Delta E \, \Delta t \simeq \frac{\hbar}{2} = \frac{h}{4\pi}$$

or, $$\Delta E = \frac{h}{4\pi \Delta t}$$

The lifetime of an electron in an excited state is of the order of 10^{-8} s, i.e., $\Delta t = 10^{-8}$ s. This implies that the excited energy levels of an atom must have a finite width rather than sharp line. So, we can say that the spectral lines have a finite width.

10. Prove that the rest mass of a photon is zero.

We know that phase velocity $v_p = \frac{\omega}{k} = \frac{\hbar\omega}{\hbar k} = \frac{E}{P}$ ...(1)

where E and P are total relativistic energy and momentum of a paticle.

Again, for relativistic particle total energy E follow the relation

$$E^2 = p^2c^2 + m_o^2 c^4$$

So from Eq. (1), $$v_p = \frac{E}{p} = \left[\frac{p^2c^2 + m_o^2 c^4}{p^2}\right]^{1/2}$$

$$= c\left[1 + \frac{m_o^2 c^2}{p^2}\right]^{1/2} \quad ...(2)$$

Since the velocity of the photon is c, so $v_p = c$

we have from Eq. (2),

$$c = c\left(1 + \frac{m_o^2 c^2}{p^2}\right)^{1/2} = c\left(1 + \frac{m_o^2 c^2 \lambda^2}{h^2}\right)^{1/2}$$

$\left[\text{According to de Broglie hypothesis } \lambda = \frac{h}{p}\right]$

which gives

$$\frac{m_o^2 c^2 \lambda^2}{h^2} = 0$$

The equation is valid only when $m_o = 0$

So the rest mass of the photon is zero.

Part 2: Descriptive Questions

1. (a) State the characteristics of black-body radiation.

   (b) Deduce Planck's law for the spectral distribution of energy in black-body radiation.

   (c) Show graphically, how the energy density versus frequency (or the wavelength) plot of a black-body radiation is changed if the temperature is increased. [WBUT 2007]

2. State clearly, explaining all the terms, the Planck's law, Rayleigh – Jeans law and Wien's displacement law for radiation. Find out the two limits at which the Planck's formula reduces to the other two. [WBUT 2006]

3. (a) State Planck's hypothesis and hence derive Planck's radiation law. [WBUT 2005]

   (b) Show that the temperature dependence in Stephan's law can be derived from Planck's radiation law. [WBUT 2008]

4. (a) Derive an expression for Compton shift in wavelength for a photon scattered from a free electron at an angle θ. [WBUT 2002, 2005]

   (b) At which angle will the shift be maximum? [WBUT 2005]

5. A photon of energy E is incident on a stationary electron target and the angle of Compton scattering of the photon is θ. Show, using non-relativistic kinetic energy that the KE of recoil of the electron is

   $$\frac{E^2(1 - \cos\theta)}{mc^2 + E(1 - \cos\theta)}$$ where m is the mass of the electron.

6. (a) Why cannot Compton effect be observed with visible light?

   (b) Derive the expression for Compton shift when a photon interacts with an electron. Show that it is impossible to transfer all the energy of the photon to the electron. [WBUT 2008]

7. Discuss how classical approaches failed to account for the spectral distribution of energy density in a black-body radiation. How did Planck overcome this difficulty? Discuss Wien's displacement law from Planck's law.

8. (a) What is Compton effect?

   (b) Why unmodified line appears in Compton scattering?

   (c) Prove that Compton shift is independent of the nature of the scatterer.

9. (a) What do you mean by matter wave? [WBUT 2002]

   (b) What is the experimental evidence in favour of de Broglie hypothesis of matter waves? [WBUT 2005]

10. (a) Describe the experiment which justifies de Broglie's hypothesis about the wave nature of particles.

11. (a) What is wave-particle duality?
(b) Show that the de Broglie wavelength λ associated with an electron of mass m and kinetic energy E_k is given by

$$\lambda = \frac{h}{\sqrt{2m\,E_k}}$$

(c) Describe the Davision and Germer experiment giving emphasis on the interpretation of the result of the experiment.

12. (a) Show that the de Broglie wavelength of a particle of mass m and kinetic energy E_k is given by

$$\lambda = \frac{hc}{\sqrt{E_k\,(E_k + 2m_o c^2)}}$$

(b) Why is quantum mechanics used in case of moving electrons while for a moving car we use newtonian mechanics? Explain. [WBUT 2005]

13. (a) Starting from de Broglie's hypothesis show that the group velocity associated with a particle is the same as the particle velocity. [WBUT 2007]
(b) Show that the de Broglie wavelength λ of electrons of charge e and energy E (in eV) is given by,

$$\lambda = \frac{h}{\sqrt{2meV}}$$

[WBUT 2005]

14. (a) State and explain Heisenberg uncertainty principle. [WBUT 2003, 2005]
(b) Show that there cannot be any existance of free electrons inside the nucleus of an atom. [WBUT 2005]

15. (a) Starting from de Broglie wave concept obtain Heisenbergs uncertainty principle.

$$\Delta x\, \Delta p_x \geq \hbar$$

(b) Give one illustration of this principle.
(c) Discuss the significance and importance of uncertainty principle.

Part 3: Numerical Problems

1. Calculate the number of photons emitted by a 10 watt source of monochromatic light having wavelength of 100 nm. **[5.03 × 18]**
2. X-rays of wavelength $\lambda = 0.124$ Å is scattered from a block of graphite. What is the wavelength of the X-rays scattered in the backward direction?
3. Calculate the maximum energy in eV, that can be transferred to an electron in a Compton experiment when the incident quanta are X-rays of wavelength 0.50 Å. **[4.7 eV]**
4. X-rays of wavelength 10^{-11}m are scattered by loosely bound electrons. Find the maximum wavelength present in the scattered rays and maximum kinetic energy of the recoil electron. **[1.485×10^{-11} m, 40.576 KeV]**
5. A γ-ray beam of wavelength 1.8×10^{-2} Å is scattered by free electrons at an angle of 90° with the incident beam. Calculate the Compton wavelength shift. **[0.0242 Å]**
6. Calculate the Compton wavelength (in Å) of an electron. [WBUT 2005] **[0.0242 Å]**

7. X-ray of wavelength λ_0 = 2.00 Å are scattered from a block of carbon. The scattered X-ray are observed at an angle of 45° to the incident beam.
   (a) Calculate the Compton shift and the wavelength of the scattered X-ray at $\phi = 45°$.
   (b) Find the fraction of energy lost by the photon in this collision. **[2.007 Å, 0.354%]**
8. Calculate the wavelength associated with an electron accelerated to a potential difference of 1.25 K eV. **[3.46×10^{-11} m]**
9. What is the de Broglie wavelength of an electron moving with a velocity, $v = \frac{3}{5}c$? **[0.0323 Å]**
10. You have 10 eV photon and 10 eV electron. Which one has shorter wavelength? Explain. **[electron]**
11. Find out the de Broglie wavelength of a neutron of kinetic energy 10 MeV?
12. Davision and Germer studied electron diffraction with nickel crystal and found a first-order peak at 65° with electron beam of 54 eV. If instead, a 216 eV beam were used, find the angle at which the corresponding peak will be seen. [WBUT 2007] **[26.9°]**
13. What is the de Broglie wavelengths of any electron which has been accelerated from rest through a potential difference of 100 V? **[1.225 Å]**
14. Calculate the minimum uncertainty in position of an electron moving with a velocity of 3×10^7 ms^{-1}. **[0.019 Å]**
15. The average time interval that elapses between the excitation of an atom and the time it radiates energy is 10^{-8} sec. Calculate the uncertainty in the energy of emitted photons and the limit of accuracy with which the frequency of the emitted radiation may be determined. **[4.14×10^{-7} eV, 10^8 cycle/sec]**
16. Calculate the minimum energy of a photon for its existence within the nucleus of diameter 10^{-14} m. **[4.9 MeV]**
17. On an average, an excited state of a system exists in that state for 10^{-11} s. What is the minimum uncertainty in the energy of an excited state? [Question Bank, WBUT] **[1.054×10^{-23} J]**

CHAPTER

10

Crystallography

10.1 INTRODUCTION

The study of the geometrical form and other physical properties of crystalline solids by using X-rays, neutron beams and electron beams constitutes the science of crystallography. Based on the nature of arrangement of microparticles, solid can be classified as crystalline solids and amorphous solids. Crystalline solids are those in which the atoms or molecules are arranged in a very regular and orderly fashion in a three-dimensional pattern. The atoms or molecules are strongly bound and arranged in a regular manner inside the crystal. Further, when a crystal breaks, the broken pieces are all having regular shape. The crystalline solids have directional properties, i.e., the physical properties like thermal conductivity, electrical conductivity have different values in different directions and therefore they are called ***anisotropic substances***. Crystalline solids may be made up of metallic crystals or non-metallic crystals. Examples of metallic crystals are copper, aluminium, silver tungsten and examples of non-metallic crystals are crystalline carbon, crystalline polymers. *Amorphous solids* have no regular structure. The amorphous solids are those in which atoms or molecules are arranged in an irregular manner and does not have any external geometrical shape, even though the atoms or molecules are strongly bound to one another. Amorphous solids have no directional properties and therefore they are known as ***isotropic substances***. The most important amorphous materials are glasses, plastics and rubbers. The structure of the crystalline solids are mainly determined by X-ray diffraction, electron diffraction and neutron diffraction, knowing the structural properties of a crystal one can determine thermal, electrical mechanical and optical properties of it.

10.2 CRYSTAL LATTICE AND LATTICE STRUCTURE

A crystal is constructed by the infinite repetation in space of identical structural units. One can replace each unit by a geometrical point. The result is a pattern of points having the same geometrical properties as the crystal. **This geometrical pattern is the crystal lattice or simply lattice.**

Lattice points Lattice points denote the position of atoms or molecules in the crystals.

Space lattice **An arrangement of an infinite array of points in the form of a two-dimensional or a three-dimensional network in a crystal is known as space lattice.** In the lattice array (or space lattice) every point has surroundings environments identical to that of every other point in the array.

10.2.1 Unit Cell

The unit cell is the smallest block or geometric figure from which the entire crystal is built up by repetition in three dimensions. If we divide the whole crystal into very small entities, we get a group of atoms in the form of a definite structure. That group of atoms of definite shape and size is known as **unit cell**. It is also known as fundamental elementary pattern of a crystal because it is repeated again and again to form the lattice structure of a crystal. **In general an unit cell may be defined as that minimum volume of a solid from which the entire crystal can be constructed by translational repetition in three dimensions.**

In order to understand the basic characteristics of a crystal structure, we show a two dimensional arrangement of the lattice points in Fig. 10.1.

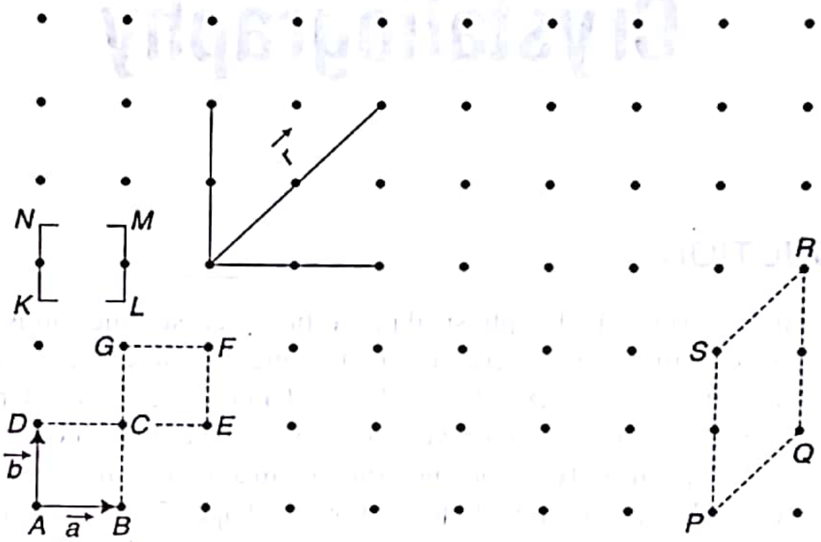


Fig. 10.1 Array of points in two dimensions.

Here we consider the case of a two-dimensional (say in x-y plane) array of atoms in Fig. 10.1. Suppose $\vec{a}$ and $\vec{b}$ are two **fundamental translational vectors** along x and y axes. If the parallelogram $ABCD$ is first translated through $\vec{a}$ and then through $\vec{b}$, the parallelogram $CEFG$ is produced. The arrangement of the lattice points in the parallelogram produced by the translations is identical with the arrangement in the intial position. It is called ***translational symmetry*** which is responsible for periodic structure of the lattice. The angle between directions of translation among a and b may be or may not be equal to 90°.

The parallelogram $ABCD$ is called the unit cell. The unit cell in a crystal lattice may be chosen in different manners. For examples in Fig. 10.1, instead of $ABCD$, we could have chosen the parallelogram $PQRS$ and $KLMN$ as unit cell.

Primitive cell The primitive cell is defined as a unit cell which has only one atom at each corner of it and no where else in Fig. 10.1, the unit cell $ABCD$ has lattice points at each of the four corners only and is called the primitive cell. So, all primitive cells are unit cells but all unit cells may or may not be primitive cells.

Non-primitive cell If the cell contains additional one or more atoms with an atom at each corner of the unit cell, these cells are called non-primitive cells. In Fig. 10.1, unit cell $PQRS$ is a non-primitive cell.

In two dimensions, choosing any point as origin, the position of any lattice point in space lattice denoted by $\vec{r}$ can be expressed in terms of translational vectors $\vec{a}$ and $\vec{b}$ as

$$\vec{r} = m\vec{a} + n\vec{b} \qquad ...(10.1)$$

In Fig. 10.1 $m = 2$ and $n = 2$

In three dimensions if $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three transnational vectors, then we can write lattice vector $\vec{r}$ as

$$\vec{r} = m\vec{a} + n\vec{b} + o\vec{c} \qquad ...(10.2)$$

where m, n and o are integers.

The parallelepiped with the sides a, b and c along these three vectors constitute the unit cell (Fig. 10.2). By repeated translations of this unit cell through integral multiples of the vector a, b and c, we get the entire crystal structure.

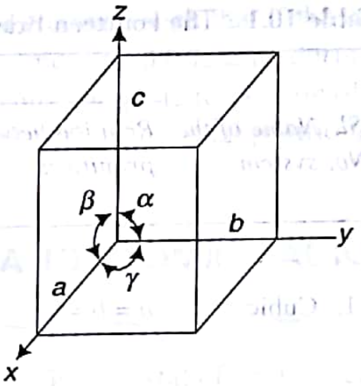


Fig. 10.2 A three-dimensional unit cell.

10.2.2 Basis or Motif

A crystal is formed by placing an atom or a group of atoms at each lattice point in a regular manner. Such an atom or a group of atoms serves as the basic building unit for the entire crystal structure and it is termed as basis or motif. The basis may be a single atom, two atoms or more. For a single atom, basis is called **monoatomic**, for two atoms, basis is called **diatomic** and so on. In case of Al, basis is monoatomic, in KCl it is diatomic, in SnO_2 it is triatomic. The crystal structure is formed when a lattice is combined with a basis (Fig. 10.3)

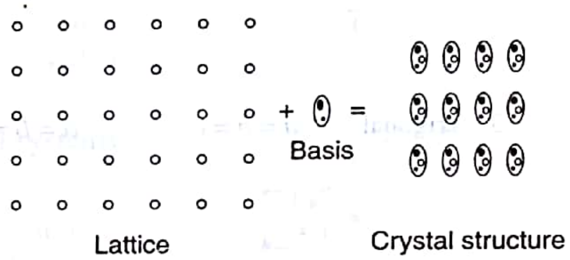


Fig. 10.3 Basis at every lattice point.

The logical relation is

Space lattice + Basis = Crystal structure

10.2.3 Bravais Lattices

In 1845, the French crystallographer Bravais first introduced the concept of the three-dimensional lattice while explaining the structure of crystal. He showed that only 14 types of space lattices are possible in the seven systems of crystals. It has been proved mathematically that there are only fourteen independent ways of arranging points in three-dimensional space such that each arrangement confarms to the defination of a space lattice. Thus the fourteen possible space lattices of the seven crystal systems are: one triclinic, two monoclinic, four orthorhombic, two tetragonal, one hexagonal, one rhombohedral and three cubic. The shapes of these units are shown in Table 10.1.

10.3 SYMMETRY ELEMENTS OF A CRYSTALLINE SOLID

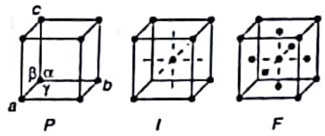
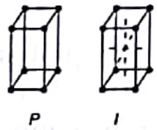
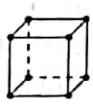
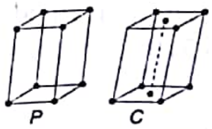
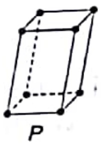
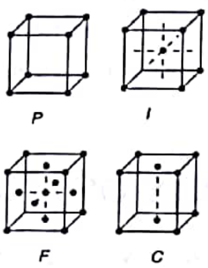
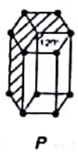
The seven crystal system are characterized by three symmetry elements: (i) the centre of symmetry, (ii) the planes of symmetry and (iii) the axes of symmetry.

A symmetry operation is an operation performed on an object or pattern which brings it to a position which is absolutely indistinguishable from the old position.

A symmetry element is an operator which performs the symmetry operation.

(i) Centre of symmetry It is a point in a crystal such that any line passing through it meets the surface of the crystal at equal distances in both directions. It is also known as the centre of inversion. Each cubic crystal has one centre of symmetry.

Table 10.1 The Fourteen Bravais Lattices in Three Dimensions

Sl. No.	Name of the system	Lattice parameters: Relation between primitives	Lattice parameters: Interfacial angles	Shape of the Bravais Lattice with Lattice symbol*	No. of possible Bravais lattices in the system	Examples
1.	Cubic	$a = b = c$	$\alpha = \beta = \gamma = 90°$		3	NaCl, Cu, Al, Cr etc
2.	Tetragonal	$a = b \neq c$	$\alpha = \beta = \gamma = 90°$		2	SnO_2, TiO_3, $NiSO_4$
3.	Trigonal	$a = b = c$	$\alpha = \beta = \gamma \neq 90°$		1	Sb, $CaSO_4$, As etc.
4.	Monoclinic	$a \neq b \neq c$	$\alpha = \beta = 90° \neq \gamma$		2	Na_2SO_4, $FeSO_4$ etc.
5.	Triclinic	$a \neq b \neq c$	$\alpha \neq \beta \neq \gamma \neq 90°$		1	[$CuSO_4$, $5H_2O$], K_2SO_4, $K_2Cr_2O_7$
6.	Ortho-rhombic	$a \neq b \neq c$	$\alpha = \beta = \gamma = 90°$		4	KNO_3, $MgSO_4$, $PbCO_3$
7.	Hexagonal	$a = b \neq c$	$\alpha = \beta = 90°$ $\gamma = 120°$		1	AgCl, Mg, Zn etc.

*p – primitve unit cell, C – base centered, I – Innenzentrierte, a German word meaning body centered F – face centered

(ii) Plane of symmetry An imaginary plane passing through a crystal such that portions on the two sides of the plane are exactly alike is known as a plane of symmetry. This is also known as bilateral symmetry.

(iii) ***Axis of symmetry*** Axis of symmetry is a line passing through the crystal such that if the crystal is rotated about this line as axis, it will present the same appearance more than one time during one complete revolution. If a cube is rotated about a vertical axis then in one complete revolution of 360° there are found to be four positions of the cube which are coincident with its original position. This rotation axis is known as an **axis of symmetry**.

10.4 ATOMIC RADIUS, CO-ORDINATION NUMBER AND ATOMIC PACKING FACTOR (APF) OF CUBIC CRYSTAL SYSTEM

Atomic radius It is defined as half the distance between nearest neighbours in a crystal of a pure element. It is denoted by the symbol r. It must be remembered that any two nearest neighbouring atoms touch each other.

Coordination Number (N) The number of equidistant nearest neighbouring atoms with respect to a particular atom in a given crystal is known as coordination number.

The coordination number for the three types of cubic crystal systems can be calculated as follows.

Simple Cubic Crystal In simple cubic crystal each corner atom is surrounded by 6 atoms at a distance a. There are 12 atoms at distances $\sqrt{2}\,a$ and so on. In Fig. 10.4a there are 4 nearest neighbours (shown as 1, 2, 3 and 4) of a particular atom 0 in its own plane and there are 2 more atoms, one directly above and the other one directly below (shown as 5 and 6).

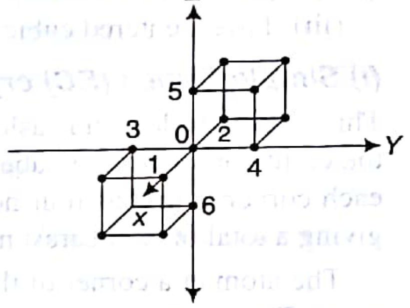


Fig. 10.4 (a) Simple cubic crystal

Hence the coordination number in this case is 6.

Face Centered Cubic Crystal In a face centered cubic crystal any corner atom O (Fig. 10.4b) is surrounded by 4 atoms at a distances $\frac{a}{\sqrt{2}}$ each. One can draw three mutually perpendicular planes through the corner atom O. In each plane, it will have 4 nearest neighbours at a distance $\frac{a}{\sqrt{2}}$. So, in three dimensions, the immediate neighbouring atoms at a distance $\frac{a}{\sqrt{2}}$ is 12.

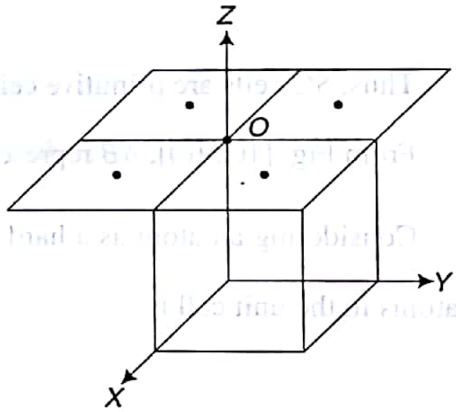


Fig. 10.4 (b) Face centered cubic structure

Thus, the coordination number in this case is 12.

Body Centered Cubic Crystal In a body centered cubic crystal, each body centered atom is surrounded by eight atoms at a distances $\frac{\sqrt{3}a}{2}$ (Fig. 10.4c). The other atoms are present at a greater distances.

Hence, the coordination number of an atom in a body centered cubic crystal is 8.

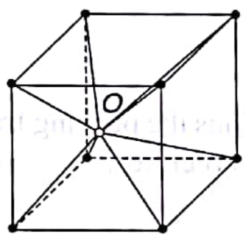


Fig. 10.4 (c) Body centered cubic structure

Atomic Packing Factor (APF) It is the ratio of the volume occupied by the atoms in a unit cell (v) to the total volume of the unit cell (V) is termed as atomic packing factor (APF). Greater the coordination number, higher is the volume of APF and more closely packed will be the crystal structure. The atomic packing factor for SC, BCC and FCC crystals are 0.52, 0.68 and 0.74 respectively.

10.5 CUBIC CRYSTAL SYSTEM

We shall now discuss briefly the crystal structures of many materials which are relatively simple. The simplest form of seven crystal systems is cubic crystal system. The cubic structure is the simplest known type of array in which the atoms take positions at the corners of the cube. There are three types of cubic crystals. These are given below:

(i) Simple Cubic (SC) crystal
(ii) Body-Centered cubic (BCC) crystal
(iii) Face Centered cubic (FCC) crystal

(i) Simple Cubes (SC) crystal

This is the simplest and easiest structure of the crystal system. In an SC crystal, there is one atom, at each of the eight corners of the cube [Fig. 10.5(a, b)]. In an SC crystal the coordination number is six. In this case, each corner atom has four neighbours in the same plane, one vertically above and one immediately below, giving a total of six nearest neighbouring atoms.

The atom at a corner of the unit cell is shared by the eight different unit cells having the common corner point. There is no atom inside the unit cell.

Therefore, the effective number of atom per unit cell $= 8 \times \frac{1}{8}$

$= 1$ atom

Thus, SC cells are primitive cells.

From Fig. [10.5(c)], AB represent the atomic radius and if a is the lattice constant, then $r = \frac{a}{2}$.

Considering an atom as a hard sphere of radius r, the volume of an atom is $\frac{4}{3}\pi r^3$. So, volume of all the atoms in the unit cell is

$$v = 1 \times \frac{4}{3}\pi r^3 \text{ and volume of the unit cells } V = a^3.$$

$$\therefore \quad \text{atomic packing factor (APF)} = \frac{v}{V} = \frac{4}{3}\frac{\pi r^3}{a^3} = \frac{\frac{4}{3}\pi r^3}{(2r)^3}$$

$$= \frac{\pi}{6} = 0.52 \text{ or } 52\%$$

Thus the packing fraction is about 52% and hence this structure is a loosely packed one. Polonium exhibits this structure.

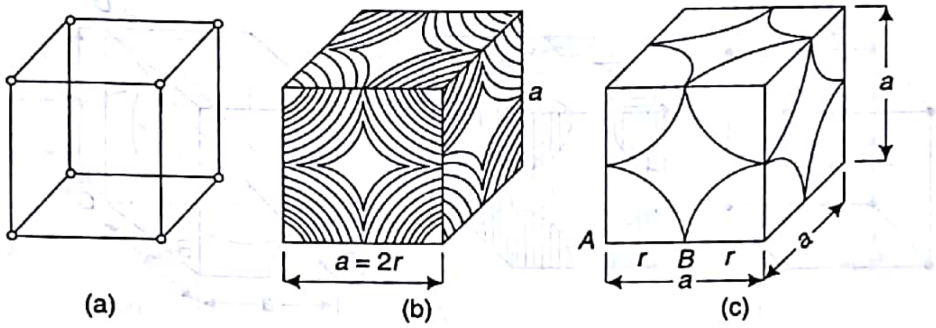


Fig. 10.5 Simple cubic crystal.

(ii) Body Centered Cubic (BCC) Crystal

The crystal structure of BCC is shown in Fig. (10.6a) which contains an atom at the centre of the cube apart from eight atoms at the corners. Each atom in this structure has eight neighbours [Fig. 10.6(a, b)]. For BCC, the coordination number is eight. Since, there are eight surrounding unit cells for any corner atom, their eight body centred atoms form the nearest neighbours for any corner atom. The number of atoms in each unit cell of this type are as follows: one centre atom and $8 \times \frac{1}{8} = 1$ corner atoms. Hence the number of atoms per unit cells = 1 + 1 = 2. Therefore, BCC cells are not primitive cells. From Fig. 10.6(c) $AD = 4r$, so we can write

$$AC^2 = AB^2 + BC^2 = a^2 + a^2 = 2a^2$$

In the triangle ACD,
$$AD^2 = AC^2 + CD^2$$
$$= 2a^2 + a^2 = 3a^2$$

or,
$$AD = \sqrt{3}\,a.$$

or,
$$4r = \sqrt{3}\,a \quad \text{or,} \quad a = \frac{4}{\sqrt{3}}r$$

The volume of an atom is $\frac{4}{3}\pi r^3$. So, the volume of all atoms in the unit cells is

$$v = 2 \times \frac{4}{3}\pi r^3 = \frac{8}{3}\pi r^3$$

Volume of all the unit cell is $V = a^3 = \left(\frac{4r}{\sqrt{3}}\right)^3 = \frac{64r^3}{3\sqrt{3}}$

$\therefore$ atomic packing factor, $\text{APF} = \frac{v}{V}$

$$= \frac{8}{3}\pi r^3 \times \frac{3\sqrt{3}}{64r^3}$$
$$= \frac{\sqrt{3}}{8}\pi = 0.68$$
$$= 68\%$$

It means that 68% space of the unit cell is occupied by the atom and the rest 32% space is void space. Elements like Fe, Na, K, Cr, Li exhibit this structure.

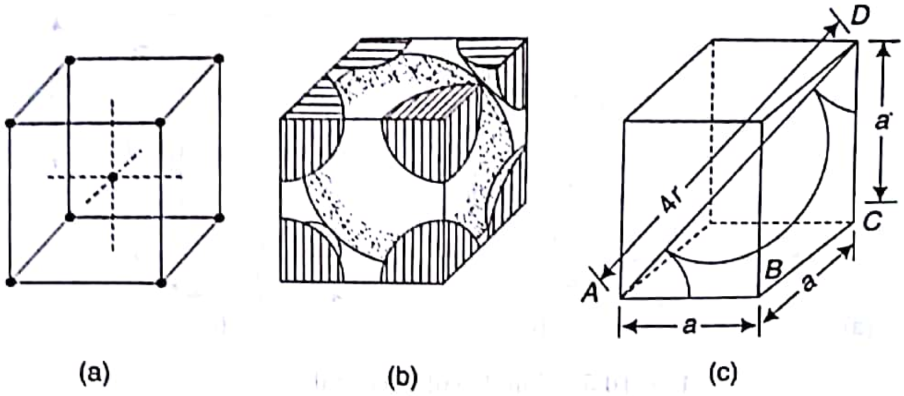


Fig. 10.6 Body Centered Cubic (BCC) crystal.

(iii) Face Centered Cubic (FCC) Crystal

The crystal structure of FCC is shown in Fig. 10.7 (a, b), which contains six atoms at the centres of each of the six faces of the cube, in addition to eight atoms at the eight corners. In an FCC crystal the coordination number is twelve. In this case, each corner atom has four neighbours atoms as has been shown in Fig. 10.7(b). One can draw three mutually perpendicular planes passing through a corner atom. So the total number of nearest neighbours is 12.

Each of the six face-centered atoms is shared by the two adjoining cubes. Hence, a total of $\frac{6}{2} = 3$ such atoms belong to the unit cell. Therefore, the effective number of atoms per unit cell of FCC is 3 + 1 = 4

Thus FCC unit cells are not primitive cells.

From Fig. 10.7(c), in the triangle ABC

$$AC^2 = AB^2 + BC^2 = a^2 + a^2 = 2a^2$$

But $AC = 4r$

Therefore $(4r)^2 = 2a^2$ or, $a = \sqrt{8}\, r$

The volume of an atom is $\frac{4}{3}\pi r^3$. So, volume of all atoms in the unit cell is

$$v = 4 \times \frac{4}{3}\pi r^3 = \frac{16}{3}\pi r^3$$

Volume of the unit cell is $V = a^3 = (\sqrt{8}\, r)^3$

$\therefore$ atomic packing factor, $$\text{APF} = \frac{v}{V} = \frac{16}{3}\frac{\pi r^3}{(\sqrt{8}\, r)^3} = \frac{\pi}{3\sqrt{2}}$$

$$= 0.74 \text{ or } 74\%$$

Thus, the packing fraction is about 74% and the rest 26% space is void space. Hence, this structure is a loosely packed one. Elements like Cu, Ag, Pb, Ca, Al etc. exhibit this structure.

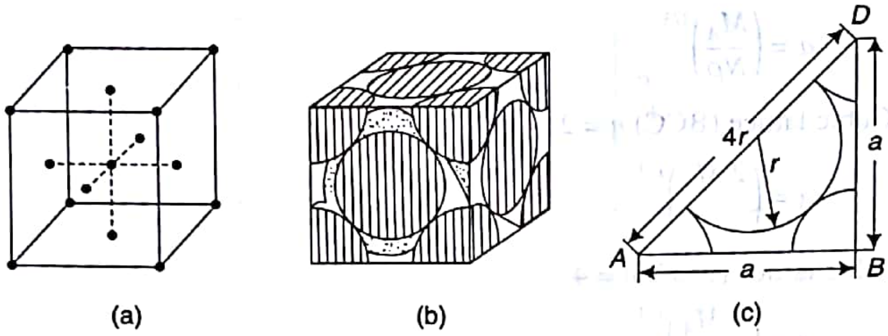


Fig. 10.7 Face Centered Cubic (FCC) crystal.

Table 10.2 Properties of Some Crystal Structures

Sl. No	*Properties*	*Simple Cube (SC)*	*Body Centered Cube (BCC)*	*Face Centered Cube (FCC)*
1.	Unit cell volume	a^3	a^3	a^3
2.	No. of atoms per unit cell	1	2	4
3.	Coordination number	6	8	12
4.	Nearest neighbouring distance (2*r*)	a	$\frac{\sqrt{3}a}{2}$	$\frac{a}{\sqrt{2}}$
5.	Density of packing (APF)	52%	68%	74%
6.	Example	Polonium	Iron, Barium, Chromium, Sodium, Tungsten	Aluminium, Copper Lead Gold

10.6 RELATION BETWEEN THE DENSITY (ρ) OF CRYSTALLINE MATERIAL AND LATTICE CONSTANT (a) OF A CUBIC LATTICE

Let us consider a cubic crystal with lattice constant a and density of the material ρ. The volume of the unit cell is equal to a^3. If n be the number of atoms (or no. of lattice points) per unit cell and M_A be the atomic weight then mass of a unit cell $= n\frac{M_A}{N}$, where N is the Avogadro's number. Again mass of each unit cell is ρa^3

So $$\rho a^3 = \frac{nM_A}{N}$$

or, $$a = \left(\frac{nM_A}{N\rho}\right)^{1/3} \qquad \text{...(10.3)}$$

This is the relation between lattice constant and density of the material of a cubic lattice.

For Simple Cubic lattice (SC), $n = 1$.

So, $$a = \left(\frac{M_A}{N\rho}\right)^{1/3} \quad ...(10.4)$$

For Body Centered Cubic lattice (BCC) $n = 2$

So, $$a = \left(\frac{2\,M_A}{N\rho}\right)^{1/3} \quad ...(10.5)$$

For Face Centered Cubic lattice (FCC) $n = 4$

So, $$a = \left(\frac{4\,M_A}{N\rho}\right)^{1/3} \quad ...(10.6)$$

10.7 NOMENCLATURE OF CRYSTAL DIRECTIONS

One basic necessity in crystal analysis is to have a system of notation for directions in a crystal. Let us indicate the direction $\overrightarrow{OP}$ in Fig. 10.8(a). The distance from O to P can be obtained by going along the x-axis a distance u times the unit distance a, then along the y-axis a distance v times the unit distance b and then in the z-direction distance w times the unit distance c. If u, v and w are the smallest integers the direction of $\overrightarrow{OP}$ is written [uvw] with square brackets. Fig. 10.8(b) gives the notation for some important direction in a cubic crystal. The square brackets are used to specify a direction with commas. The digits in square brackets indicate the indices of the direction.

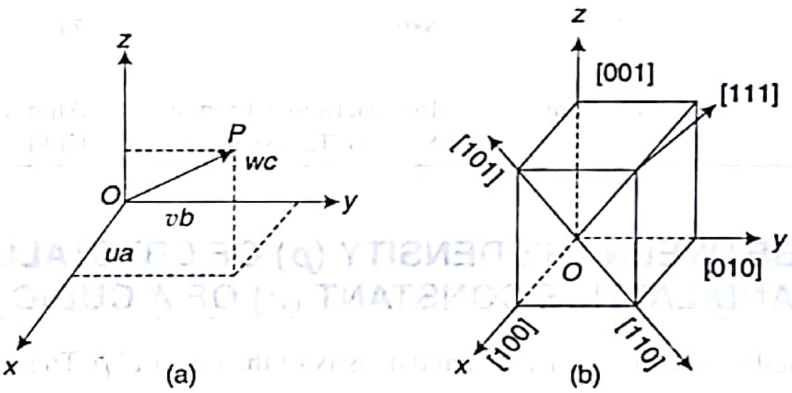


Fig. 10.8 Crystal directions in 3D. (a) Direction of $\overrightarrow{OP}$ and (b) Some important directions in cubic lattice.

A negative index is indicated by a bar over the digit. For example positive x-axis has indices of [100] (read one zero zero not one hundred) whereas negative x-axis has indices of [100].

10.8 MILLER INDICES

It is often necessary to identify planes passing through a space lattice. Orientation of planes in crystal may be described in terms of their intercepts on the three axes. For example, plane. ABC [Fig. 10.9(a)] has intercepts of 4 axial units on x-axis and 4 axial unit on y-axis and 2 axial unit on z-axis. Thus, the coordinates of intercepts are (4, 4,2). However, as suggested by Miller, it is more useful to describe the orientation of a plane by the reciprocal of its numerical perimeters rather than by its linear parameters. These reciprocals when converted to whole numbers are called **Miller indices.** Hence, Miller indices of the crystal plane ABC of Fig. 10.9(a) are $\left(\frac{1}{4} : \frac{1}{4} : \frac{1}{2}\right)$ or (1 1 2).

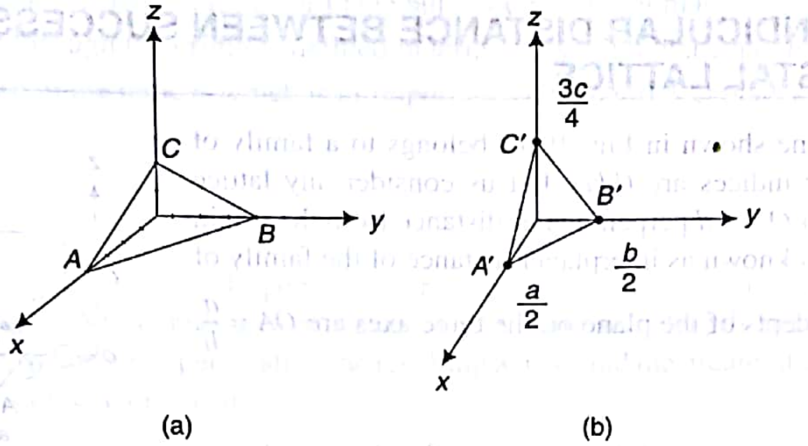


Fig. 10.9 Lattice plane intercepting the three crystallographic axes.

Now, consider plane $A'\ B'\ C'$ [Fig. 10.9(b)] whose intercepts expressed in axial units are $\frac{a}{2}, \frac{b}{2}, \frac{3c}{4}$. Its numerical parameters are $\frac{1}{2}, \frac{1}{2}$ and $\frac{3}{4}$. Hence, its Miller indices which are nothing but reciprocal of its numerical parameters $\left(\frac{1}{\frac{1}{2}} : \frac{1}{\frac{1}{2}} : \frac{1}{\frac{3}{4}}\right)$ or $\left(2, 2, \frac{4}{3}\right)$ or (6 6 4) or (3 3 2). In a cubic crystal (100) is equal to the set of its six faces (100), (010), (001), (100), (010) and (001). The symbols for a family of parallel planes is <*hkl*>. So we can define Miller indices of a crystal plane as the reciprocals of the intercepts of a plane on the crystallographic axes when it is reduced in form of smallest numbers.

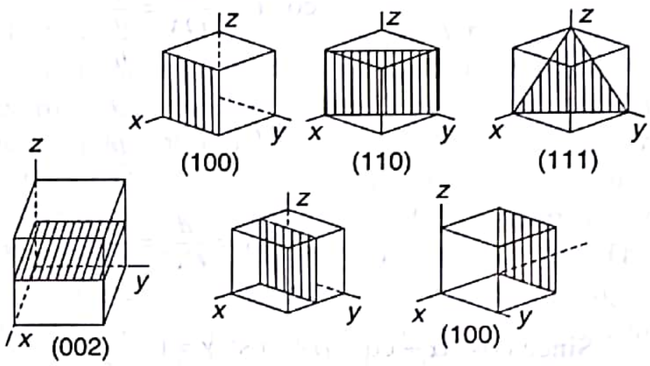


Fig. 10.10 Shows some important planes in a cubic crystal system.

10.8.1 Procedure for Finding Miller Indices

(a) Find the intercepts of the plane on the three coordinate axes prefererably crystallographic axes. Let these are x_1, y_1, z_1 along x, y, and z axes respectively.

(b) Then express them in terms of the axial units. Let these be $x_1 = ua$, $y_1 = vb$ and $z = wc$.

(c) Next take the numerical parameters of the plane i.e. $u : v : w$. The numerical parameters must be in whole numbers.

(d) Find the ratio of their reciprocals $\left(\frac{1}{u} : \frac{1}{v} : \frac{1}{w}\right)$.

(e) Convert these reciprocals into whole numbers by multiplying each with their LCM to get the smallest whole number.

(f) Denote the results in the form of (*hkl*) which called Miller indices of the plane.

10.9 PERPENDICULAR DISTANCE BETWEEN SUCCESSIVE PLANES IN A CRYSTAL LATTICE

Suppose that the plane shown in Fig. 10.11 belongs to a family of planes whose Miller indices are (hkl). Let us consider any lattice point O as origin. Let $ON = d$ perpendicular distance from the origin to the plane, which is known as interplaner distance of the family of the planes. The intercepts of the plane on the three axes are $OA = \frac{a}{h}$, $OB = \frac{b}{k}$ and $OC = \frac{c}{l}$.

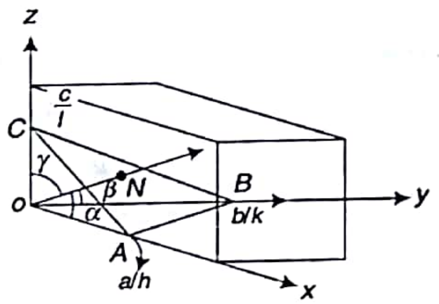


Fig. 10.11 Interplaner distance of the family of the planes.

Let the direction cosines of ON be $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ where

$$\cos\alpha = \frac{d}{OA} = \frac{d}{\frac{a}{h}}$$

$$\cos\beta = \frac{d}{OB} = \frac{d}{\frac{b}{k}}$$

$$\cos\gamma = \frac{d}{OC} = \frac{d}{\frac{c}{l}}$$

Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ ...(10.7)

or, $$\frac{d^2}{\left(\frac{a}{h}\right)^2} + \frac{d^2}{\left(\frac{b}{k}\right)^2} + \frac{d^2}{\left(\frac{c}{l}\right)^2} = 1$$

or, $$d^2\left[\frac{h^2}{a^2} + \frac{k^2}{b^2} + \frac{l^2}{c^2}\right] = 1$$

or, $$d = \frac{1}{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2} + \frac{l^2}{c^2}\right)^{1/2}} \quad \text{...(10.8)}$$

This equation gives the distance between successive plane in a crystal lattice.

For cubic lattice $a = b = c$

$\therefore$ $$d = \frac{a}{\sqrt{h^2 + k^2 + l^2}} \quad \text{...(10.9)}$$

10.10 IMPORTANT FEATURES OF MILLER INDICES

(i) All equally spaced parallel planes have the same index number (hkl)

(ii) A crystal plane parallel to one of the crystallographic axis has intercepts at infinity and index of zero for that axis.

(iii) Miller indices do not define a particular plane but a set of parallel planes.

(iv) A plane passing through the origin is defined in terms of parallel plane having non-zero intercepts.

(v) It is only the ratio of the indices which is of importance i.e., the (6 2 2) planes are the same as (3 1 1) planes.

10.11 X-RAYS

In 1895, W Roentgen found that a highly penetrating radiation of unknown nature is produced when fast electrons impinge on matter. These rays are em waves of very short wavelength and known as X-rays. The faster the original electrons, the more penetrating the resulting X-rays and the greater the numbers of electrons, the greater the intensity of the X-ray beam.

10.12 ORIGIN OF X-RAYS

When high-speed (about 10% of the velocity of light) electrons from the cathode of an X-ray tube strike the target, the electrons penetrate the atoms of the target and remove an electron from the inner shells by collision. This is illustrated in Fig. 10.12(a), where an electron is knocked out of the K shell. We know that K, L, M .... electron shells correspond to $n = 1, 2, 3$ .... When a tightly bound electron is missing in the inner-most K shell, the vacant position of that electron is occupied by an electron from next higher level and emits a photon of energy $h\nu$. Such X-rays produce K lines [Fig. 10.12(b)]. Now an electron from M shell jumps into L shell vacancy and as a consequence, there is emission of X-rays of another frequency known as L lines. This process continues until the uppermost shell is reached where an electron jumping gives rise to visible light. So, it is possible for a single atom to emit X-rays of different wavelengths. The continuous X-ray spectrum is obtained due to slowing down of high speed electrons as they pass close to the nuclei of the atoms within the target of the X-ray tube. As the electron passes through the atom, it is attracted by the positive charge of the nucleus and is deflected in its path. The process is shown in the Fig. 10.13. The electron is deaccelerated during its deflection in the strong field of the nucleus. The kinetic energy lost during this retardation is transformed into electromagnetic radiation in the form of X-rays of continuously varying wavelengths. These X-rays form continuous spectrum when analyzed by Bragg spectrometer. Let the electron is slowed down to a velocity v' from the initial velocity v. Considering the conservation of energy, we get

$$\frac{1}{2}mv^2 - \frac{1}{2}mv'^2 = h\nu \qquad ...(10.10)$$

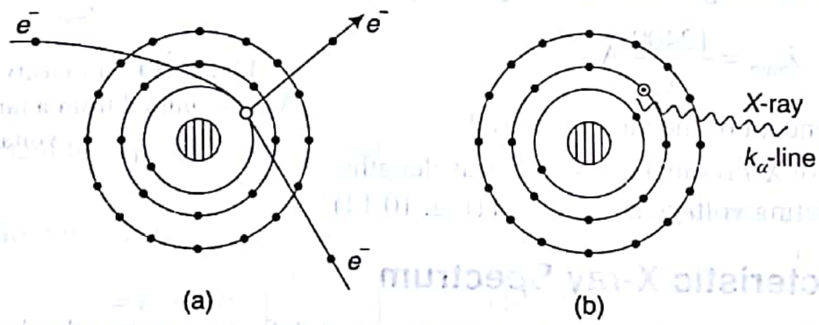


Fig. 10.12 Transition of an electron from L-shell to K-shell at the time of radiation.

The highest frequency is possible when the electron is completely stopped by the atom. In this case, we have

$$\frac{1}{2}mv^2 = h\nu_{max} \quad ...(10.11)$$

Minimum limiting wavelength λ_{min} is obtained when the entire kinetic energy of the bombarding electron is converted to X-ray energy.

If V be the accelerating potential, then

$$\frac{hc}{\lambda_{min}} = eV$$

$$\therefore \quad \lambda_{min} = \frac{hc}{eV} \quad ...(10.12)$$

This is known as Duane Hunt law

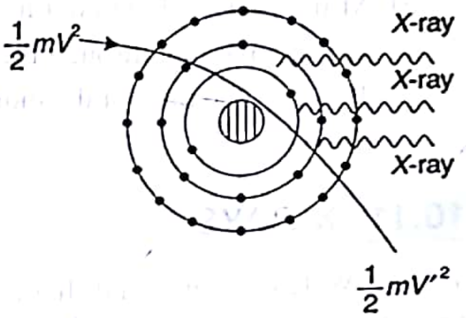


Fig. 10.13 Origin of continuous spectrum.

10.13 X-RAY SPECTRUM

There are two types of X-ray spectrum

(a) Continuous X-ray spectrum (b) Characteristic (or line) X-ray spectrum

10.13.1 Continuous X-ray Spectrum

The X-ray spectrum which consists of all possible wavelengths between a maximum and minimum values and depending solely on the applied voltage across the X-ray tube called as continuous X-ray spectrum.

Some of the important features of the continuous X-rays:

(i) The continuous spectrum produced because the de-acceleration of high velocity electrons while passing near the positively charged nucleus of an atom of the target material.

(ii) The shortest wavelength limit can be given by

$$\lambda_{min} = \frac{12400}{V} \text{ Å}$$

(iii) λ_{min} is independent of the target material.

(iv) The intensity of X-rays increases at all wavelengths as the accelerating voltage is increased (Fig. 10.14)

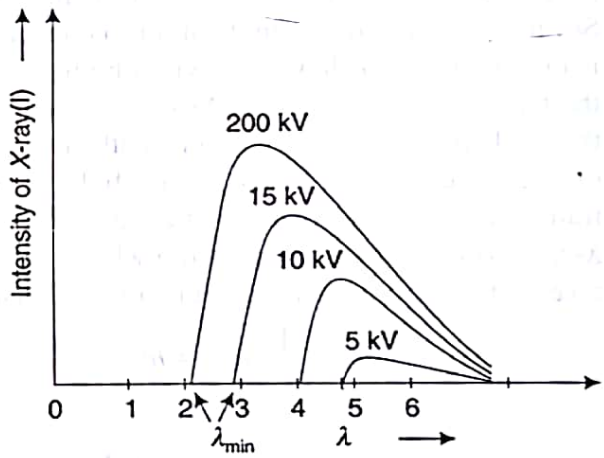


Fig. 10.14 Intensity (*I*) vs. wavelength of X-rays emitted from a target element at different voltages.

10.13.2 Characteristic X-ray Spectrum

The X-ray spectrum which has a line spectrum consisting of definite wavelengths depending on the characteristics of the target material and superimposed on the continuous spectrum is called the characteristic X-ray spectrum (Fig. 10.15)

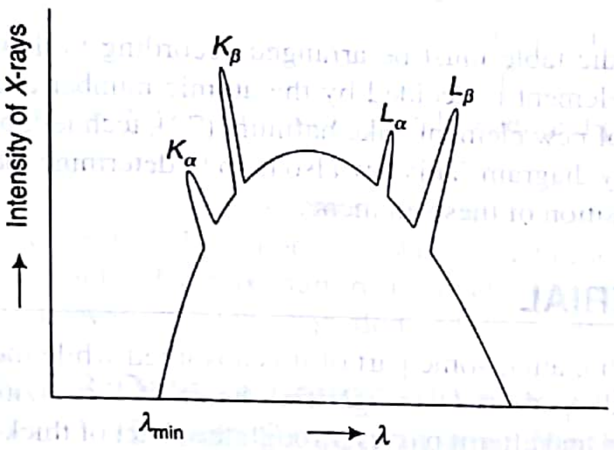


Fig. 10.15 Characteristic *X*-ray spectrum

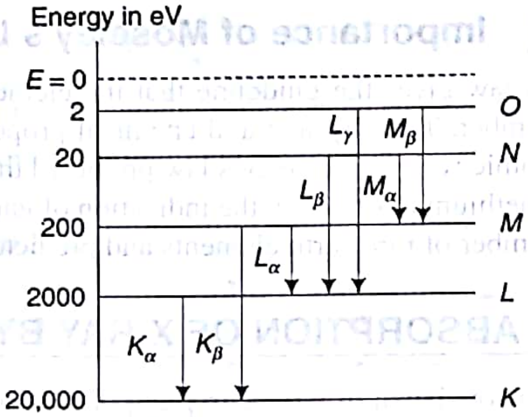


Fig. 10.16 Formation of *K*-series, *L*-series and *M*-series of characteristic *X*-rays spectrum in energy level diagram

Some important features of the characteristic *X*-rays spectrum are as follows:

(i) It is produced when innermost electrons are removed from the atom of the target and the position of the electron is taken by the electron of outer orbits.

(ii) In this case discrete spectral lines constitute *K*-series; *L*-series and *M*-series etc. (Fig. 10.16)

(iii) *K*-series constitute hard *X*-rays (*X*-ray with high penetrating power) and *L*, *M* series form soft *X*-rays (*X*-ray with low penetrating power)

(iv) There is a regular shift towards shorter wavelength in the *K*-spectrum as atomic number of the target is increased.

10.14 MOSELEY'S LAW

Moseley carried out study of characteristic *X*-rays spectra of various metallic elements and observed that if the square root of the frequency ν of the most intense spectral line i.e., *K* series line is plotted against the atomic number *Z*, then a straight line graph is obtained (Fig. 10.17) for each and every element. The graph is known as Moseley diagram. The relationship between frequency and atomic number *Z* as

$$\nu \propto (Z-b)^2$$

or
$$\sqrt{\nu} = a\,(Z-b) \qquad \text{...(10.13)}$$

where *a* and *b* are constant for a particular series but varies from one series to another.

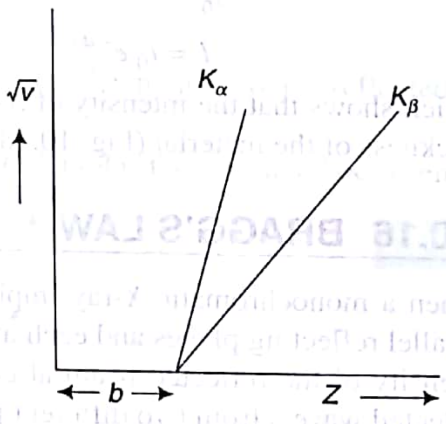


Fig. 10.17 Variation of $\sqrt{\nu}$ with atomic number (*Z*).

An exact form of Moseley's law is

$$\frac{1}{\lambda} = R(Z-b)^2\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right) \qquad \text{...(10.14)}$$

where n_1 and n_2 are principal quantum numbers and *R* is the Rydberg constant.

10.14.1 Importance of Moseley's Law

Moseley's law gives the guideline that the elements in periodic table must be arranged according to their atomic number. The physical and chemical properties of an element is decided by the atomic number and not the atomic weight. Moseley's law predicted the existence of new elements like hafnium (72), technetium (43), promethium (61) etc. by the indication of gaps in Moseley diagram. This law also help to determine the atomic number of rare earth elements and predicted proper position of these elements.

10.15 ABSORPTION OF X-RAY BY A MATERIAL

When a narrow beam of monochromatic X-rays passes through matter, some part of it is absorbed while the rest is transmitted. If a sheet of any substance is interposed in the path of a homogeneous beam of X-rays, its intensity decreases. Let I_0 and I be the intensity of X-rays before and after it passes through the sheet of thickness x. We know that the rate of decrease of intensity with thickness $\left(\text{i.e., } -\frac{dI}{dx}\right)$ is proportional to I, then

$$-\frac{dI}{dx} \propto I \quad \text{or,} \quad \frac{dI}{dx} = -\mu I \qquad ...(10.15)$$

where μ is called linear absorption coefficient of the absorber.

or, $$\frac{dI}{I} = -\mu\, dx$$

Integrating both sides, we have

$$\int_{I_0}^{I} \frac{dI}{I} = -\mu \int_0^x dx$$

$$\log \frac{I}{I_0} = -\mu x.$$

$$I = I_0\, e^{-\mu x} \qquad ...(10.16)$$

which shows that the intensity of X-rays decreases exponentially with thickness of the material (Fig. 10.18)

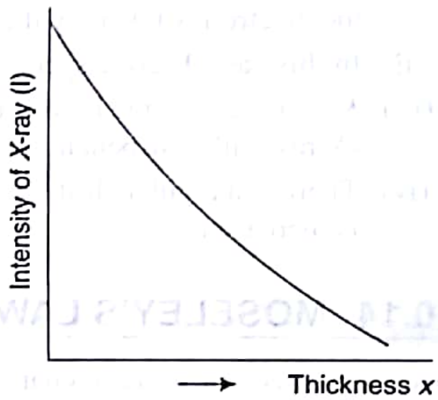


Fig. 10.18 Variation of intensity with thickness of the material.

10.16 BRAGG'S LAW

When a monochromatic X-ray impinges upon the atoms in a crystal lattice, the crystal acts as a series of parallel reflecting planes and each atom acts as a source of scattering radiation of the same wavelength. The intensity of the reflected beam at certain angles will be maximum when the path difference between two reflected wave's from two different planes is an integral multiple of λ.

10.16.1 Derivation of Bragg's Law

Consider a set of parallel planes of atom at a spacing d between two successive planes (Fig. 10.19). Let a beam of monochrometic X-ray of wavelength λ be incident on the first plane at a glancing angle θ. Let us consider two parallel rays PQR and $P'Q'R'$ in the beam reflected by two atoms Q and Q'. The ray $P'Q'R'$ has a longer path then the ray PQR. To get the path difference, perpendicular QT and QS on $P'Q'$ and $Q'R'$ respectively. Then the path difference between the two rays is

$$\delta = TQ' + Q'S$$
$$= d \sin \theta + d \sin \theta$$
$$= 2d \sin \theta \quad ...(10.17)$$

The two rays will reinforce each other and produce maximum intensity if

$$2d \sin \theta = n\lambda \quad ...(10.18)$$

where $n = 1, 2, 3, ...$ For $n = 1$ we get first order and for $n = 2$ we get second order maximum respectively. The equation $2d \sin \theta = n\lambda$ is known as Bragg's equation and represents Bragg's law.

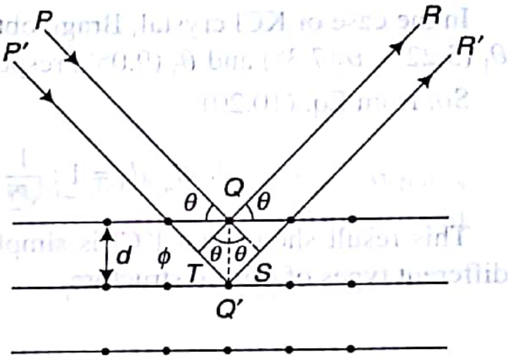


Fig. 10.19 Bragg's reflection of X-rays from the atomic plane.

10.16.2 Determination of Lattice Constant: (Bragg Spectrometer Method)

There are various methods of determining the lattice constants, such as law spot, powdered crystal, Bragg X-ray spectrometer, rotating crystal etc. Here we describe the Bragg spectrometer method.

The schematic arrangement of Bragg's spectrometer is shown in Fig. 10.20. X-ray from X-ray tube are narrowed to obtain a fine pencil of beam by passing through slits S_1 and S_2. The beam is now allowed to fall on a crystal C mounted on a circular turn table of the spectrometer and it position can be recorded by the vernier V capable of moving along the circular scale S. The reflected beam then passes through slits S_3 and enters ionization chamber through a narrow window w made of aluminium. The ionization chamber is simply a container for gas or vapour with two electrodes. One of the electrodes is connected to the positive terminal of H.T. battery and the negative terminal connected to quadrant electrometer. The turn table and ionization chamber are linked together in such a way that when the turn table rotates through an angle θ, the ionization chamber turns through 2θ. The X-rays entering the ionization chamber ionize the gas which deflect the electrometer and gives the intensity of the reflected X-ray.

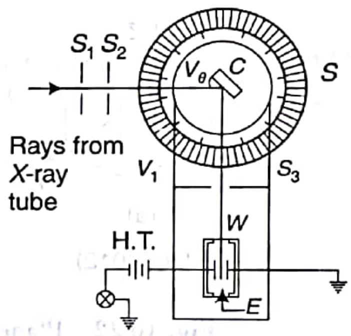


Fig. 10.20 Bragg's spectrometer

The ionization current is measured for different values of glancing angle θ. A plot is then obtained between θ and the ionization current [Fig. 10.21]. For certain values of θ, the intensity of ionization current increases abruptly.

Again from Bragg's law

$$2d \sin \theta = n\lambda \quad ...(10.19)$$

The Bragg's angles at which reflection has a maximum are determined with the help of this apparatus. Let us consider the first order reflection from three different planes of the crystal are θ_1, θ_2 and θ_3. From Eqn. (10.19) we get

$$\lambda = 2d_1 \sin \theta_1 = 2d_2 \sin \theta_2 = 2d_3 \sin \theta_3$$

or, $$d_1 : d_2 : d_3 = \frac{\lambda}{\sin \theta_1} : \frac{\lambda}{\sin \theta_2} : \frac{\lambda}{\sin \theta_3} \quad ...(10.20)$$

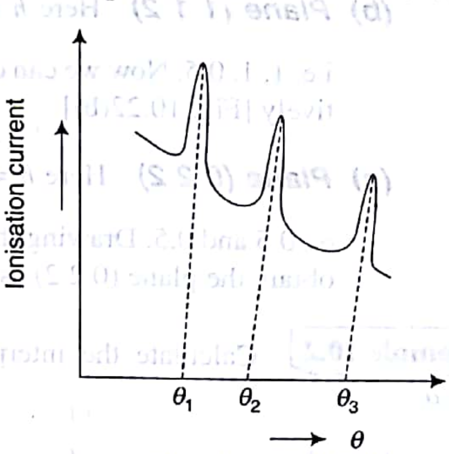


Fig. 10.21 Variation of ionization current with glancing angle θ

In the case of KCl crystal, Bragg obtained the maxima of reflected X-rays ($\lambda = 0.71$Å) at glancing angles θ_1 (5.22°), θ_2(7.3°) and θ_3 (9.05°) respectively using three different reflecting planes.

So, from Eq. (10.20)

$$d_1, d_2, d_3 = 1 : \frac{1}{\sqrt{2}} : \frac{1}{\sqrt{3}} \quad \text{...(10.21)}$$

This result shows that KCl is simple cubic crystal. In this way Bragg's law can be utilized to analyze different types of crystal structure.

Worked-out Examples

Crystallography

Example 10.1 Draw the planes (0 1 2), (1 1 2), (0 2 2) in a FCC structure.

Sol.

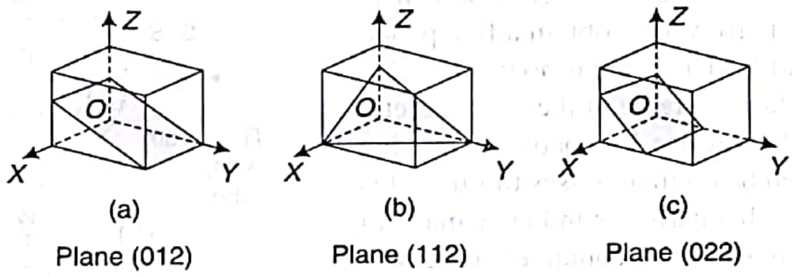


Fig. 10.22 Plane (0 1 2), Plane (1 1 2) and plane (0 2 2) in a cubic crystal system

(a) Plane (0 1 2) Here $h = 0$, $k = 1$, $l = 2$ the reciprocals are $\frac{1}{0}, \frac{1}{1}$ and $\frac{1}{2}$ i.e. ∝, 1 and 0.5. Drawing the plane with intercepts ∝, 1 and 0.5 along x, y and z-axis respectively, we obtain the plane (0 1 2) as shown in Fig. 10.21(a)

(b) Plane (1 1 2) Here $h = 1$, $k = 1$, and $l = 2$. The reciprocals of h, k and l are $\frac{1}{1}, \frac{1}{1}$ and $\frac{1}{2}$; i.e. 1, 1, 0.5. Now we can draw the plane with intercepts 1, 1 and 0.5 along x, y and z axes respectively [Fig. 10.22(b)]

(c) Plane (0 2 2) Here $h = 0$, $k = 2$ and $l = 2$. The reciprocals of h, k and l are $\frac{1}{0}, \frac{1}{2}$ and $\frac{1}{2}$ i.e., ∝, 0.5 and 0.5. Drawing the plane with intercepts ∝, 1 and 0.5 along x, y, z axes respectively, we obtain the plane (0 2 2) as shown in Fig. 10.22(c)

Example 10.2 Calculate the interplaner spacing d of a planes (1 1 1) in a simple cubic lattice of side a. [WBUT 2003, 2004, 2007]

Sol. Interplaner spacing $$d_{hkl} = \frac{1}{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2} + \frac{l^2}{c^2}\right)^{1/2}}$$

Here $h = k = l = 1$ and $a = b = c$

$$\therefore \quad d_{111} = \frac{1}{(3/a^2)^{1/2}} = \frac{a}{\sqrt{3}}$$

Example 10.3 The distance between (100) planes in a body-centered cubic structure is 0.232 nm. What is the size of the unit cell? What is the radius of the atom? [WBUT 2006]

Sol. If the radius of the atom is r of a BCC crystal then $r = \frac{\sqrt{3}a}{4}$, where a is lattice constant

For BCC $\quad d_{100} = \frac{a}{2}$

or,
$$a = 2d_{100} = 2 \times 0.232$$
$$= 0.464 \text{ nm}$$

and radius
$$r = \frac{\sqrt{3}a}{4} = \frac{\sqrt{3}}{4} \times 0.464 = 0.20 \text{ nm.}$$

Example 10.4 Copper has FCC structure and the atomic radius is 0.1278 nm. Calculate its density and the interplanar spacing for (3 2 1) planes. Taking the atomic weight of copper as 63.5. [WBUT 2005]

Sol. For a cubic crystal, the lattice constant

$$a = \left(\frac{n M_A}{\rho N}\right)^{1/3}$$

For FCC $n = 4$ (number of atom per unit cell)

$M = 63.5$

$N = 6.023 \times 10^{23}$/g mole

ρ = density

Again for FCC $\quad a = \frac{4}{\sqrt{2}} r$

r = atomic radius = 0.1278 nm

$= 0.1278 \times 10^{-7}$ cm

So, density
$$\rho = \frac{nM}{Na^3}$$

$$= \frac{4 \times 63.5}{6.023 \times 10^{23} \times \left(\frac{4 \times 0.1278 \times 10^{-7}}{\sqrt{2}}\right)^3} \text{ gm cm}^{-3}$$

$$= 8.926 \text{ g/cm}^3$$

The interplaner spacing for (3 2 1)

$$d = \frac{a}{(h^2 + k^2 + l^2)^{1/2}} = \frac{\frac{4}{\sqrt{2}} \times 0.1278 \times 10^{-7}}{\sqrt{3^2 + 2^2 + 1^2}} = 0.0966 \text{ nm}$$

Example 10.5 In a crystal, a lattice plane cuts intercepts of $2a$, $3b$ and $6c$ along the axes, where a, b and c are primitive vectors of the unit cell. Determine the Miller indices of the plane.

Sol. We know that if the given plane cut intercepts pa, qb and rc along the three axes, we have

$$pa : qb : rc = 2a : 3b : 6c$$

$$\therefore \quad p : q : r = 2 : 3 : 6$$

$$\therefore \quad \frac{1}{p} : \frac{1}{q} : \frac{1}{r} = \frac{1}{2} : \frac{1}{3} : \frac{1}{6} = 3 : 2 : 1$$

So Miller indices of the plane are (3 2 1)

Example 10.6 Find the interplanar distance of (1 1 0) plane and (1 1 1) plane of Nickel crystal. The radius of Nickel atom is 1.245 Å.

Sol. Nickel has FCC structure. So lattice constant

$$a = \frac{4}{\sqrt{2}} r \quad r \text{ is the radius of Nickel}$$

$$= \frac{4 \times 1.245}{\sqrt{2}} = 3.52 \text{ Å}$$

Therefore, the interplanar distance of (1 1 0) plane is

$$d_{110} = \frac{3.52}{\sqrt{1^2 + 1^2 + 0}} = 2.49 \text{ Å}$$

The interplanar distance of (1 1 1) plane is

$$d_{111} = \frac{3.52}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{3.52}{\sqrt{3}} = 2.03 \text{ Å}$$

Example 10.7 If the potential difference applied across an X-ray tube is 12.4 kV and the current through it is 2 mA. Calculate (i) the number of electrons striking the target per second, and (ii) the speed with which they strike it.

Sol. (i) Current $I = 2 \text{ mA} = 2 \times 10^{-3}$ A

$= ne$ where n is the number of electrons striking the target per second

so $$n = \frac{I}{e} = \frac{2 \times 10^{-3}}{1.6 \times 10^{-19}} = 1.25 \times 10^{16}$$

(ii) If v is the speed with which electrons strike the target

$$\frac{1}{2} mv^2 = Ve$$

$$v = \sqrt{\frac{2Ve}{m}} = \sqrt{\frac{2 \times 1.6 \times 10^{-19} \times 12.4 \times 10^3}{9.1 \times 10^{-31}}}$$

$$= 6.6 \times 10^7 \text{ m/s}$$

Example 10.8 If the energy levels of K, L, M state of Pt are 78 KeV, 12 KeV and 3 KeV respectively, then calculate the wavelengths of k_α and k_β lines emitted from Pt. ($h = 6.626 \times 10^{-34}$ JS; $C = 3 \times 10^8$ m/s). [WBUT 2008]

Sol. Energy of K level, $E_K = 78 \text{ KeV} = 78000 \times 1.6 \times 10^{-19}$ J

L level $E_L = 12 \text{ KeV} = 12000 \times 1.6 \times 10^{-19}$ J

M level $E_M = 3 \text{ KeV} = 3000 \times 1.6 \times 10^{-19}$ J

So, wavelength of k_α line

$$\lambda_\alpha = \frac{hc}{E_K - E_L} = \frac{6.63 \times 10^{-34} \times 3 \times 10^8}{(78000 - 12000) \times 1.6 \times 10^{-19}} = 0.188 \text{ Å}$$

Wavelength of k_β line $\lambda_\beta = \dfrac{hc}{E_K - E_M} = \dfrac{6.63 \times 10^{-34} \times 3 \times 10^8}{(78000 - 3000) \times 1.6 \times 10^{-19}}$

$$= 0.165 \text{ Å}$$

Example 10.9 The first order reflection from the plane of NaCl is obtained at an angle $2\theta = 20°$ with incident beam. If $d = 2.82$ Å, calculate the wavelength of X-ray used.

Sol. From Bragg's equation

$$2d \sin \theta = n\lambda$$

For $n = 1$ $2d \sin \theta = \lambda$

Here $2\theta = 20°$, $\theta = 10°$ and $d = 2.82 \times 10^{-10}$ m

$$\lambda = 2 \times 2.82 \times 10^{-10} \sin 10°$$

$$= 9.791 \times 10^{-11} \text{ m} = 0.979 \text{ Å}$$

Example 10.10 A beam of X-rays of wavelengths 0.842 Å is incident on a crystal at a glancing angle of 8°35′, when first order Bragg's reflection occurs. Calculate the glancing angle for the third order reflection. [WBUT 2002, 2007]

Sol. From Bragg's equation $2d \sin \theta_n = n\lambda$ ...(1)

For 1st order reflection, $n = 1$, $\theta_1 = 8°35'$, $\lambda = 0.842 \times 10^{-10}$ m

So, the interplaner distance

$$d = \frac{n\lambda}{2 \sin \theta_1} = \frac{0.842 \times 10^{-10}}{2 \times \sin 8°35'} = 2.8 \text{ Å}$$

For third order $n = 3$, from Eq. (1)

$$2 \times 2.8 \times 10^{-10} \times \sin \theta_3 = 3 \times 0.842 \times 10^{-10}$$

or, $$\theta_3 = 26.8° = 26° 48'$$

Example 10.11 Can X-rays be diffracted from a single slit of width 0.1 mm? Explain. [WBUT 2005]

Sol. The wavelength of X-ray is in between 1 Å to 100 Å i.e., 10^{-7} mm to 10^{-5} mm respectively. So, the wavelength of X-rays is not comparable to the width of the single slit. Hence X-rays can't be diffracted by a single slit of width 0.1 mm.

Example 10.12 The energy gap in Ge is 0.72 eV. Find the maximum wavelength of a radiation which can cause transition from the valance band to conduction band. [WBUT 2007]

Sol. We know that band gap energy

$$E_g = \frac{hc}{\lambda_{max}}$$

or,
$$\lambda_{max} = \frac{hc}{E_g} = \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{0.72 \times 1.6 \times 10^{-19}} \text{ m} = 17.276 \times 10^{-7} \text{ m} = 0.017 \text{ Å}$$

Example 10.13 The spacing of a planes in a crystal is 1.2 Å and the angle for the first-order Bragg's reflection is 30°. Determine the energy of the *X*-rays beam in eV. [WBUT 2005]

Sol. From Bragg's equation $2d \sin \theta = n\lambda$

Here $n = 1$ and $\theta_1 = 30°$ $d = 1.2$ Å

So, the wavelength of the incident beam,

$$\lambda = \frac{2d \sin \theta}{n} = \frac{2 \times 1.2 \times \sin 30°}{1} = 1.2 \text{ Å}$$

The energy of the incident *X*-ray beam

$$E = \frac{hc}{\lambda} = \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{1.2 \times 10^{-10} \times 1.6 \times 10^{-19}} \text{ eV}$$

$$= 10.35 \times 10^3 \text{ eV} = 10.35 \text{ KeV}.$$

Example 10.14 Electrons are accelerated by 344 volts and are reflected from a crystal. The first order reflection maximum occurs when glancing angle is 30°. Determine the spacing of the crystal.

Sol. Wavelength $\lambda = \frac{h}{mv} = \frac{h}{\sqrt{2mVe}}$

Here $m = 9.1 \times 10^{-31}$ kg, $V = 344$ V, $e = 1.6 \times 10^{-19}$ C

So,
$$\lambda = \frac{6.62 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 344 \times 1.6 \times 10^{-19}}}$$

$$= 0.66 \times 10^{-10} \text{ m}$$

According to Bragg's law

$$2d \sin \theta = n\lambda$$

Here $n = 1$, $\theta = 30°$ and $\lambda = 0.66 \times 10^{-10}$ m

So,
$$d = \frac{0.66 \times 10^{-10}}{2 \sin 30°} = 0.66 \times 10^{-10} \text{ m} = 0.66 \text{ Å}$$

Example 10.15 K_α radiation if molybdenum ($z = 42$) has a wavelength of 0.72 Å. Find the wavelength of the corresponding radiation of copper ($z = 29$).

Sol. From Moseley's law $v = a\,(z - b)^2$

or, $$\frac{c}{\lambda} = a\,(z-b)^2$$

$\therefore$ $$\frac{1}{\lambda} \propto (z-1)^2 \quad [\text{From } k_\alpha \text{ line } b = 1]$$

$\therefore$ $$\frac{\lambda_{cu}}{\lambda_{mo}} = \frac{(Z_{mo}-1)^2}{(Z_{cu}-1)^2} = \frac{(42-1)^2}{(29-1)^2} = \frac{(41)^2}{(28)^2}$$

or, $$\lambda_{cu} = 0.72 \times \left(\frac{41}{28}\right)^2 = 1.54\text{ Å}$$

Example 10.16 The linear absorption coefficient for Al for X-rays having $\lambda = 0.32$ Å is 1.62 cm^{-1}. Find the thickness of the absorber needed to cut down the intensity of the beam to 1/20 of the initial value.

Sol. If I_o be the initial intensity of the X-ray beam is reduced to I in traversing a distance x in absorber then

$$I = I_0\, e^{-\mu x} \quad \text{where } \mu \text{ is the linear absorption co-efficient.}$$

Now $$\frac{I}{I_0} = e^{-\mu x}$$

Here $$\frac{I}{I_0} = \frac{1}{20}, \quad \text{so} \quad \frac{1}{20} = e^{-\mu x} = e^{-1.62x}$$

or, $$e^{1.62\,x} = 20$$

or, $$1.62\,x = 2.3026 \log 20$$

$\therefore$ $$x = \frac{2.3026 \times 1.301}{1.62} = 1.85\text{ cm.}$$

Example 10.17 The X-ray analysis of crystal is made with monochromatic X-rays of wavelength 0.58 Å. Bragg's reflections are obtained at an angles of (i) 6.45° (ii) 9.15° (iii) 13°. Calculate the interplanar spacing of the crystal.

Sol. From Bragg's law,

$$\frac{d}{n} = \frac{\lambda}{2\sin\theta}$$

(i) $$\frac{d}{n} = \frac{0.58}{2 \times \sin 6.45°} = 2.568\text{ Å} \qquad ...(1)$$

(ii) $$\frac{d}{n} = \frac{0.58}{2 \times \sin 9.15°} = 1.817\text{ Å} \qquad ...(2)$$

(iii) $$\frac{d}{n} = \frac{0.58}{2 \times \sin 13°} = 1.288\text{ Å} \qquad ...(3)$$

It is clear from Eqs. (1) and (3) that the value of d/n in Eq. (1) is almost twice to that of Eq. (3). This shows that angles 6.45° and 13° represent the first and second order reflection maxima from one set

of parallel planes. The spacing may be obtained either by putting $n = 1$ in Eq. (1) or $n = 2$ in Eq. (3). Hence $\frac{d}{1} = 2.568$ Å which is equal to the interplanar spacing of the crystal.

Now the glancing angle (9.15°) used in Eq. (2) refers to some other set of reflecting planes. Their interplanar spacing is given by $\frac{d}{1} = 1.817$ Å.

Review Exercises

Part 1: Multiple Choice Questions

1. The fundamental building block whose repetition in space generates a crystal is termed as
   (a) primitive cell (b) unit cell (c) non-primitive cell (d) none of these
2. Primitive cell is equal to a
   (a) unit cell (b) unit cell in the case of simple cubic lattice
   (c) unit cell in the case of BCC lattice (d) unit cell in the case of FCC
3. Lattice plus basis form the
   (a) lattice structure (b) crystal structure (c) unit cell (d) none of these
4. In case of simple cubic structure, coordination number is
   (a) 5 (b) 6 (c) 2 (d) 1
5. Which of the following element shows simple cubic structure?
   (a) Na (b) Ca (c) Polonium (d) None of these
6. FCC lattice unit cell which consists of atoms has
   (a) 2 atoms (b) 1 atom (d) 4 atoms (d) 5 atoms
7. A crystal is equivalent to
   (a) n-dimensional grating (b) 3 dimensional grating
   (c) 1 dimensional grating (d) 2 dimensional grating
8. In case of FCC structure, coordination number is
   (a) 8 (b) 12 (c) 5 (d) 6
9. Among the cubic lattices, which has the closest packing?
   (a) FCC (b) SC (c) BCC (d) DC
10. The nearest neighbour distance in the case of BCC structure is [WBUT 2008]
    (a) $\frac{a\sqrt{3}}{3}$ (b) $\frac{a\sqrt{2}}{3}$ (c) $\frac{2a}{\sqrt{3}}$ (d) $\frac{2a}{\sqrt{2}}$
11. Assuming that the atoms in a crystal are spheres of equal size and touching each other, it can be shown that the atomic radius of BCC is equal to [WBUT 2007]
    (a) $\frac{a}{\sqrt{2}}$ (b) $\sqrt{3}\,\frac{a}{4}$ (c) $\sqrt{3}\,\frac{a}{2}$ (d) $\frac{a}{2\sqrt{2}}$

12. The primitives of crystal are 1.2 Å, 1.8 Å, 2Å along the three axes where a plane with Miller indices (2 3 1) cut intercept 1.2 Å along *X*-axis. What will be the lengths of intercept along *Y* and *Z* axis? [WBUT 2008]

(a) 1.5 Å and 5 Å (b) 1.9 Å and 2.5 Å (c) 1.2 Å and 4Å (d) 1.5 Å and 4 Å

13. The number of bravais lattices are

(a) 12 (b) 14 (c) 13 (d) 10

14. The interplanar spacing of (111) planes in a simple cubic lattice of side a is

(a) $\frac{a}{2}$ (b) $\frac{a}{\sqrt{3}}$ (c) $\frac{a}{\sqrt{2}}$ (d) a

15. The atomic packing factor for SC structure is

(a) 52 % (b) 75 % (c) 74 % (d) 68 %

16. The coordination number in case of NaCl structure is

(a) 8 (b) 6 (c) 4 (d) 2

17. The atomic packing factor for FCC structure is

(a) 74 % (b) 68 % (c) 52% (d) none of these

18. The Miller indices are used for

(a) identifying crystal direction (b) notation of crystal planes
(c) notation of different crystals (d) identifying the density of packing

19. In NaCl crystal, the unit cell contains

(a) 4 molecules (b) 6 molecules (c) 8 molecules (d) none of these

20. Miller indices of a plane which cut intercepts of 2, 3 and 4 units along the three axes are [WBUT 2006]

(a) (6, 4, 3) (b) (2, 3, 4) (c) (3, 2, 1) (d) (2, 3, 2)

21. The planes (111) and (222) are

(a) parallel to each other (b) perpendicular to each other
(c) antiparallel to each other (d) none of these

22. (111) planes of cubic crystal is [WBUT 2005]

(a) perpendicular to the *X*-axis (b) perpendicular to the *Y*-axis
(c) perpendicular to the *Z*- axis (d) none of these

23. The atomic packing factor for BCC structure is

(a) 74 % (b) 68 % (c) 52 % (d) none of these

24. Origin of continuous *X*-ray is due to the process of [WBUT 2007]

(a) ionization (b) inner orbital transition
(c) bremsstrahlung (d) none of these

25. Which of the following wavelengths falls in the *X*-ray range? [WBUT 2006]

(a) 1 nm (b) 100 nm (c) 0.001 nm (d) 1000 nm

26. The production of characteristic *X*-rays can be considered as

(a) inverse of photoelectric effect (b) inverse of pair production
(c) Compton effect (d) none of these

27. If λ_L and λ_K are wave length of L and K X-rays respectively, then [WBUT 2005]
    (a) $\lambda_L < \lambda_K$ (b) $\lambda_L > \lambda_K$ (c) $\lambda_L = \lambda_K$ (d) $\lambda_K = 4\lambda_L$
28. When two parallel rays of X-rays of wavelength λ, are incident at an angle θ on a crystal with lattice separation d, constructive interference would be observed when (n is an integer) [WBUT 2006]
    (a) $n\lambda = 2d \sin \theta$ (b) $n\lambda = d \sin \theta$ (c) $n\lambda = d \sin 2\theta$ (d) $n\lambda = 2d \sin 2\theta$
29. The line spectra are
    (a) superimposed on the continuous spectra (b) not superimposed on the continuous spectra
    (c) lies under the continuous spectra (d) none of these

Answers

1. (b)	2. (b)	3. (b)	4. (b)	5. (c)	6. (c)	7. (b)	8. (b)
9. (a)	10. (a)	11. (b)	12. (c)	13. (b)	14. (b)	15. (a)	16. (b)
17. (a)	18. (b)	19. (a)	20. (a)	21. (a)	22. (d)	23. (b)	24. (c)
25. (a)	26. (a)	27. (b)	28. (a)	29. (a)			

Short Questions with Answers

1. **What are crystallographic directions?**

   The crystallographic direction is a line joining any two points of the space lattice related to the crystal structure.

2. **Define (i) lattice points, (ii) space lattice, (iii) unit cell, (iv) primitive cell, and (v) non-primitive cell.**

   Refer to Articles (10.2) and (10.2.1).

3. **What are Bravais lattices?**

   It has been proved mathematically that there are only fourteen independent ways of arranging points in three dimensional space such that each arrangement confirms to the definition of a space lattice. Thus the fourteen possible space lattices of the seven crystal systems are called Bravais lattices.

4. **What are 2D and 3D space lattices?**

   2D space lattice is defined as an infinite array of points in two-dimensional space in which every point has the same environment with respect to all other points. In two dimensions, choosing any point as origin, the position of any lattice point in space lattice denoted by $\vec{r}$ can be expressed in terms of translational vectors a and b as

   $$\vec{r} = m\vec{a} + n\vec{b}$$

   where m and n are arbitrary integers. In Fig. 10.1 $m = 2$, $n = 2$. *3D space lattice* is defined as infinite array of points in three dimensional space in which every point has the same environment with respect to all other points.

   In three dimensions, if $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three transnational vectors along x, y and z axes, respectively then we can write lattice vector $\vec{r}$ as

   $$\vec{r} = m\vec{a} + n\vec{b} + o\vec{c}$$

   where m, n and o are integers.

5. **Why production X-ray is called inverse of photoelectric effect?**
   In case of production of continuous X-ray, kinetic energy of an electron is converted into the energy of the emitted photon. Similarly, in photo electric effect, the energy of incident photon is absorbed by an electron through inelastic collision and emission of electron takes place from the surface of a metal. So, some energy of incident photon is converted into the kinetic energy of the electron. Hence we can say that production of X-ray as the inverse process of photoelectric effect.
6. **What is white X-rays?**
   We know that there are two types of X-rays. One has a line spectrum and is called characteristic X-ray. The other has a continuous spectrum and is known as white or continuous X-rays. White X-rays are produced by the slowing down of the high-speed electrons in their motion through a substance. This process is also known as bremsstrahlung.
7. **Why are X-ray used in crystal diffraction?**
   The crystal constitutes a three-dimensional grating of periodic array of atoms. The inter-atomic spacing in a crystals is of the order of 1Å. So, to observe diffraction in crystal, wavelength of radiation used should be of the same order of interatomic spacing.
   Since, the wavelength of X-rays (≃ 0.1 Å to 100 Å) are comparable with interatomic spacing, so X-rays are used for diffraction in crystals.

Part 2: Descriptive Questions

1. (a) Define the terms unit cell, space lattice and basis.
   (b) What are the differences between crystalline and amorphous solid?
2. (a) Define primitive unit cell. Are all unit cells are primitive?
   (b) What do you mean by Bravais lattice? How many Bravais lattices are there in three dimensions?
3. Describe simple crystal system with necessary diagrams.
4. Define atomic packing factor. Calculate it for BCC and FCC structure.
5. What are Miller indices and what is their importance?
6. Find the distance between adjacent planes in crystal lattice.
7. Show that in cubic crystal of side a, the interplaner spacing between consecutive parallel planes of Miller indices (hkl) is [WBUT 2005, 2007]

$$d_{hkl} = \frac{a}{\sqrt{h^2 + k^2 + l^2}}$$

8. Find out the packing fraction for FCC structure. Find out the expression for the separation between the planes with Miller indices h, k, l and lattice constant a for a simple cubic structure.
   [WBUT 2008]
9. (a) Find the Miller indices of a plane which intercepts at a, $b/2$, $3c$ in a simple cubic unit cell.
   [WBUT 2004]
   (b) Draw the planes and directions denoted by (100) (110) and (102) of a cubic structure.
10. Calculate the atomic packing fraction and atoms per unit cell in crystals having (i) SC (ii) BCC (iii) FCC structure considering the atoms as hard sphere. [WBUT 2004]

11. Write a short note on Bravais lattice and Miller indices. [WBUT 2002]
12. Write a short note on X-ray diffraction and Bragg's law. [WBUT 2003]
13. (a) Distinguish between continuous and characteristic X-ray spectra.
    (b) Explain and derive Bragg's law of X-ray diffractions from a crystal. [WBUT 2005]
14. Describe the origin of characteristic X-rays.
15. If μ is the linear absorption coefficient of X-rays, prove that $\mu = \frac{0.693}{x_{1/2}}$ where $x_{1/2}$ is the half value of thickness of the substance. [WBUT 2004]
16. Draw continuous X-ray spectra for two different applied voltages. What will happen when the applied voltage is sufficiently high?

Part 3: Numerical Problems

1. Sodium chloride crystallizes in FCC structure. The density of NaCl is 2180 kg/m^3. If the atomic weight of sodium and chlorine are 23 and 35.5 respectively. Calculate the distance between two adjacent atoms. **[2.815Å]**
2. A crystal of NaCl mounted on a Bragg spectrometer reflects the K_α line of X-rays at an angle of 60° in the first order. Given the following data, calculate the wavelength of the X-ray K_α line.
   [Density (ρ) = 2.17 g/cc, mol. wt. of NaCl (M) = 58.63]
   [*Hints* $d = \left(\frac{M}{2N\rho}\right)^{1/3}$, Glancing angle = 90° – angle of reflection = 90 – 60 = 30°] **[2.82Å]**
3. A powder pattern is obtained for lead with radiations of $\lambda = 1.54$ Å. The (220) reflection is observed at Bragg angle $\theta = 32°$. What is the lattice parameter of the lead? Assume that given reflection is the first order reflection? **[4.1×10^{-10} m]**
4. Determine the spacing between (i) (100) planes, (ii) (110) planes, and (iii) (111) planes in a NaCl crystal having the lattice constant $a = 5.64$ Å. [5.64 Å, 3.39 Å, 3.26Å]
5. The potential difference across in X-ray tube is 10^5 volt. What is the maximum frequency emitted X-rays? **[2.4×10^{19} Hz]**
6. In an X-ray diffraction experiment, the second-order glancing angle was 30°. Calculate the glancing angle for the first order and third order. **[$\theta_1 = 14° 29'$, $\theta_3 = 48° 33'$]**
7. A X-ray tube operates at 20 kV. Find
   (a) the maximum speed of the electrons striking the anticathode
   (b) the shortest wavelength of X-ray produced

APPENDIX

A

Angular Concept

What is an Angle?

The answer of this easy question is not that easy. Let us try to understand what does it mean. The concept of an angle does not exist in one-dimensional case. In two dimensional case, by angle we mean separation of two intersecting lines. To be more specific – an angle is one sense of separation between two intersecting lines which remains same if even the lines are extended upto infinity (Fig. A1.1).

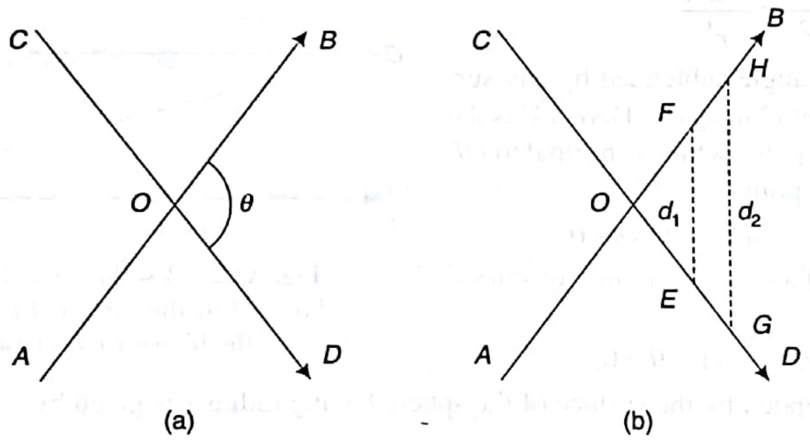


Fig. A1.1 Separation of two intersecting lines (*a*) in terms of angle (θ), (b) in terms of the distance between two equidistant points on two lines from the point of intersection.

If one wants to express the separation in terms of distance one can follow the method given below:

Let us select two pairs of equidistant points which are equidistant from the point of intersection O. For one pair of points E and F the distance is d_1 and for the other pair of points G and H it is d_2, where $d_1 \neq d_2$ but the angular separation θ between the two lines remains the same throughout.

The unit of measurement of angle is radian (symbol is ^{c} or rad) in all systems of measurement like second which is the unit for time in all systems of measurement. If the angle between two intersecting lines is 90° or 100^g (g stands for grade), then in radian one can express it as $\frac{\pi^c}{2}$ or $\frac{\pi}{2}$ rad. Radian is called a pseudounit as it does not have any representation in dimensional analysis.

Solid Angle As has been stated above the counter part of two dimensional angle (planar angle or simply angle) is the solid angle in three dimension. It is usually denoted by the Greek letter Ω. **It is the cone that an object subtends at a point in space.** It is a measure of how big an object appears to an observer looking from the said point in space. For example, a small object, which is nearer to the observer could subtend the same solid angle as a large object which is far away from the observer. The solid angle is proportional to the surface area S, of a projection of that object onto a sphere centred at the point where the observer lies, divided by the square of the radius, R, of the said sphere or it is exactly equal to the surface area, S, of the aforesaid projection of an object on to a sphere of unit radius centered at the point where the obsever lies. Symbolically, one can represent it by the following equation:

$$\Omega = K\frac{S}{R^2}$$

where K is a constant of proportionality. A solid angle is related to the surface of a sphere in the same way as an ordinary angle or planar angle is related to the circumference of a circle. If the proportionality constant is chosen to be unity, then the unit of solid angle in the SI system is steradian (denoted by sr). Thus, the solid angle of a sphere measured from a point in its interior is $4\pi\, sr$, and the solid angle subtended at the centre of a cube by one of its six surfaces is one-sixth of that (i.e., $4\pi/6\, sr$) or $2\pi/3\, sr$.

In general, one can define a solid angle as follows (Fig. A1.2):

$$d\Omega = \frac{dA'}{r^2} = \frac{dA\cos\theta}{r^2}$$

or,
$$d\Omega = \frac{dA'}{r^2} = \frac{\bar{r}\cdot d\bar{A}}{r^3}$$

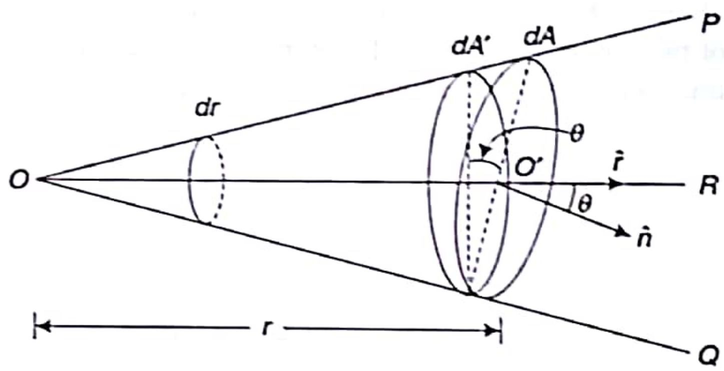


Fig. A1.2 A solid angle $d\Omega$ at the point O subtended by the surface dA' which is normal to the bisector OR of angle $\angle POQ$

where $d\Omega$ is the solid angle subtended by any surface area dA at the point O in space. Here dA' is the projection of dA on the plane which is normal to OR and passes through the poin O'

i.e., $dA' = dA\cos\theta.$

For a spherical surface whose radius of curvatuae is $OO' = r$

$$dA' = 4\pi r^2 \quad \text{and} \quad \theta = 0,$$

so the solid angle subtended by the surface of the sphere having radius r is given by

$$\Omega = \int_S \frac{dA\cos\theta}{r^2} = 4\pi\, sr.$$

APPENDIX

B

Coherence of Light

Coherence is the predictable correlated motion of two waves when there is a fixed amplitude and phase relationship between them. If two light waves, which are coming from two different sources are coherent, then these two sources are also said to be coherent. In other words, coherence is related to definite phase relationship between two waves (or sources) at different points of space and time. For a source to be coherent, it must be able to emit radiations of single frequency or the frequency spread must be very small. And also the wave-front spreading from the source must be able to maintain a constant shape. Thus we can categorize coherence into two types – temporal coherence and spatial coherence.

(a) Temporal Coherence

The temporal coherence implies the possibility of predicting the phase and the amplitude at a certain space point at different time instants. An ideal monochromatic wave represented by a simple harmonic function extends between $-\infty$ and $+\infty$ in time at a fixed point in spce. The amplitude remains constant while the phase varies linearly with respect to time. So, one can predict its amplitude and phase at any time. Such a wave is claimed to be **completely temporally coherent**. But **a real light source emits light in short pulses called wavetrains.** When an excited atom goes to a lower state or to the ground state, it emits a pulse during a time of the order of $\Delta t = 10^{-8}$ s. There will be no phase relationship between the pulses coming out of different atoms because various atoms of a source emit radiations in a random manner. Yet, at a given point of space one can predict the phase and amplitude at two different points of time if the same wavepulse still passes through that point. Thus, Δt is the longest time span over which one can make such prediction. For the time span Δt, the light is said to be coherent and the time $\Delta t = \tau_c$ **is known as coherence time.**

In order to understand coherence time, let us consider Young's double slit experiment as shown in Fig. B1.1.

The pattern of interference observed around the point P at time t is due to the superposition of the two waves emanating from S_1 and S_2 at points of time $t - r_1/c$ and $t - r_2/c$ respectively, where the variables r_1 and r_2 are the distances S_1P and S_2P respectively.

Now, if

$$\frac{r_2 - r_1}{c} << \tau_c \qquad \text{...(B1.1)}$$

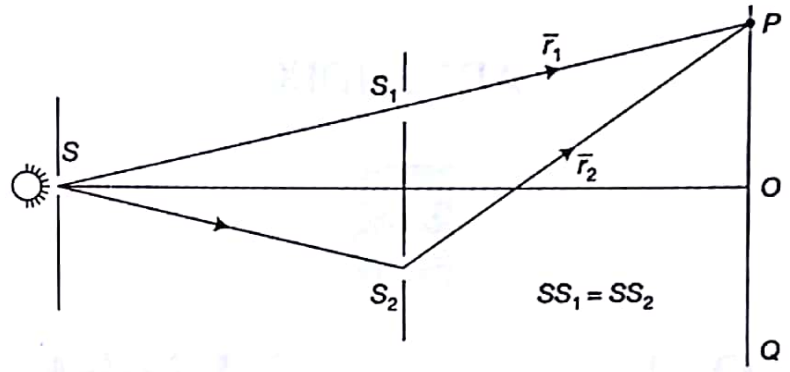


Fig. B1.1 Young's double slit experiment. The interference pattern at point P and around at time t is due to light waves emanating from S_1 and S_2 at time $t - r_1/c$ and $t - r_2/c$.

then the two waves which are arriving at P from S_1 and S_2 will have a definite phase relationship. As a result an interference pattern of good contrast will be obtained. And on the contrary, if the path difference $(r_2 - r_1)$ is sufficiently large as compared to τ_c, then

$$\frac{r_2 - r_1}{c} >> \tau_c \qquad \text{...(B1.2)}$$

and the waves arriving at P from the slits S_1 and S_2 will not have any fixed phase relationship and any interference pattern will not be seen. But the central fringe will have a good contrast if $r_1 = r_2$. The length of the wavepulse or the coherence length L_c is defined as

$$L_c = \tau_c \times C \quad \text{[where } c \text{ is speed of light]} \qquad \text{...(B1.3)}$$

The temporal coherence can be related to the line width. By Fourier analysis of the wavetrains of finite duration, one can show that a wavetrain is equivalent to a large number of harmonic waves having their frequencies within a certain interval of frequency $\Delta\nu$ about a central frequency ν. The frequency spead or line bandwidth $\Delta\nu$ is related to the coherence time τ_c by the following equation,

$$\Delta\nu = \frac{1}{\tau_c} \qquad \text{...(B1.4)}$$

Thus, narrow bandwidth indicates that the coherence time is long. For a wave to be perfectly monochromatic $\Delta\nu = 0$ and $\tau_c = \infty$. Thus, **a monochromatic wave is completely temporally coherent.**

(b) Spatial Coherence

The spatial coherence refers to the phase relationship between two light waves present at two space points at the same point of time. If there is a point source S then two equidistant points P and Q (which are not along the same line with S) will always have a definite phase relationship [Fig. B1.2].

But if one gradually increases the size of the source S then after sometime the points P and Q will fail to maintain definite phase relationship. **The lateral dimension of the source upto which the radiations from the source remain coherent determines the spatial coherence.** That is, the spatial coherence depends on the size of the source.

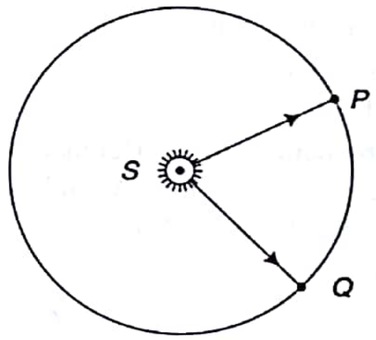


Fig. B1.2 Points P and Q which are equidistant from the source S can maintain definite phase relationship.

In order to understand the concept of spatial coherence let us consider the Young's double slit experiment again with two independent point-sources S and S' which do not have any phase relationship.[Fig. B1.3]. Let S_1 and S_2 be two slits separated by a distance $2d$ from one another. Each of the sources produces its own interference pattern on screen $S_1' S_2''$. Let SS_1 be equal to SS_2 and hence S_1O is equal to S_2O. So, the source S produces a maximum around the point O on the screen $S'_1S''_2$. But the intensity at O due to the other source S' depends on the optical path difference $S'S_2 - S'S_1$. From the figure, we can write

$$S'S_2 = [D^2 + (d+l)^2]^{\frac{1}{2}} \approx D\left[1 + \frac{1}{2}\left(\frac{d+l}{D}\right)^2\right]$$

and
$$S'S_1 = [D^2 + (d-l)^2]^{\frac{1}{2}} \approx D\left[1 + \frac{1}{2}\left(\frac{d-l}{D}\right)^2\right]$$

$$\therefore \quad S'S_2 - S'S_1 = 2\frac{dl}{D} \qquad \text{...(B1.5)}$$

Now, if $S'S_2 - S'S_1 = \dfrac{\lambda}{2}$

then, $$\frac{2dl}{D} = \frac{\lambda}{2}$$

Hence $$l = \frac{1}{2} \times \frac{\lambda D}{2d} \qquad \text{...(B1.6)}$$

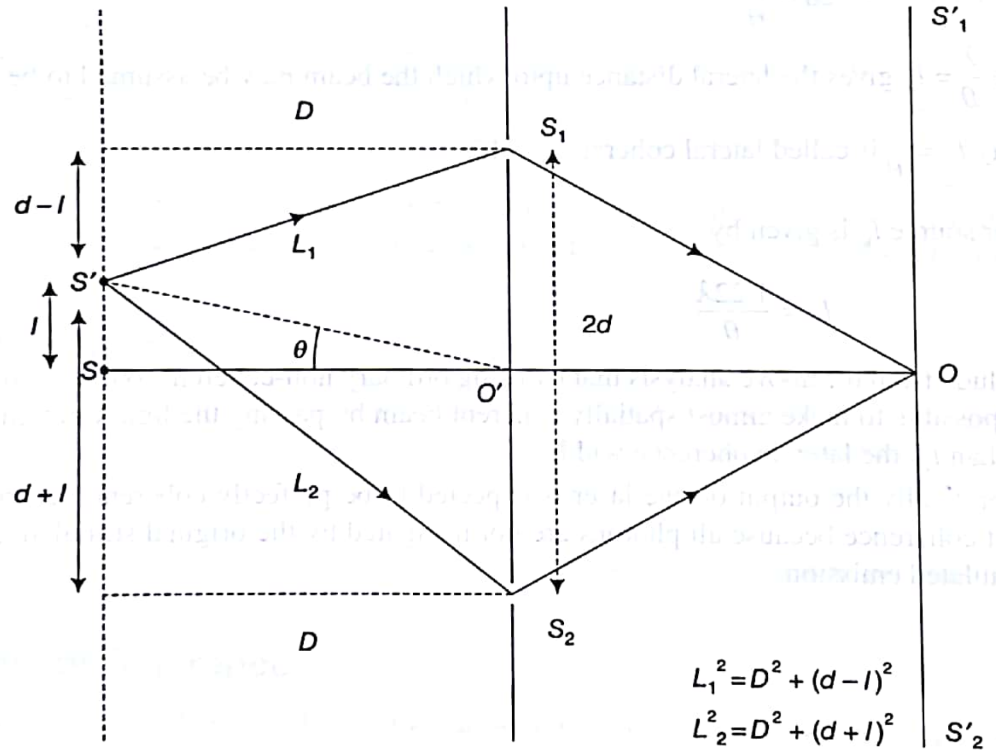


Fig. B1.3 Young's double slit experiment with two independent point sources S and S' at a distance l

The source S' produces a minimum at 0 whereas the source S produces a maximum. Eventually no interference pattern can be observed for the separation l of the sources S and S' given by the Eq. (B1.6).

Now, if we get an extended source having linear dimension $\frac{\lambda D}{2d}$ instead of $\frac{1}{2} \times \frac{\lambda D}{2d}$ (= l), then for every point on the source we may get an independent point at a distane $l\left(=\frac{1}{2} \times \frac{\lambda D}{2d}\right)$. The above analysis indicates that no interference pattern can be obtained. Good interference fringes can be obtained so long as the linear dimension l of the source is such that

$$l << \frac{\lambda D}{2d} \qquad \text{...(B1.7)}$$

Equivalently, for a source of width l definite phase relationship can exist between S_1 and S_2 if the separation between them is given by

$$2d << \frac{\lambda D}{l} \qquad \text{...(B1.8)}$$

Since the angle subtended by the source SS' at $0'$ is $\frac{l}{D} = \theta$, the above condition (Eq. (B1.8)) can be expressed as

$$2d << \frac{\lambda}{\theta} \qquad \text{...(B1.9)}$$

Thus, if the distance between the slits S_1 and S_2 is increased from zero the interference fringes vanish when

$$2d \sim \frac{\lambda}{\theta} \qquad \text{...(B1.10)}$$

The distance $\frac{\lambda}{\theta} = l_w$ gives the lateral distance upto which the beam may be assumed to be spatially coherent. The quantity $l_w = \frac{\lambda}{\theta}$ is called lateral coherence width.

For a circular source l_w is given by

$$l_w = \frac{1.22\lambda}{\theta} \qquad \text{...(B1.11)}$$

We can conclude from the above analysis that by using ordinary non-coherent source such as electric bulb or the sun, it is possible to make almost spatially coherent beam by passing the light through a small hole of size much less than l_w, the lateral coherence width.

Though, theoretically the output of one laser is expected to be perfectly coherent, but in reality, it does not show perfect coherence because all photons are not instigated by the original stimulating photon which initiates the stimulated emission.

APPENDIX

C

Relativistic Concept

(Special Theory of Relativity)

Introduction

Einstein put forward the special theory of relativity in 1905. This theory had revolutionized physical concepts during the twentieth century. Let us present a brief discussion of the special theory of relativity here as the concepts of dependence of mass on velocity, mass energy equivalence and energy-momentum relation are included in the syllabus of quantum physics. Physics is the 'science of measurements' of various physical quantities. The special theory of relativity reveals that the measurements depend on the state of motion of both the observer and the objects that are being observed. When the idea of relativity is incorporated into mechanics, one gets a new subject called relativistic mechanics. And in relativistic mechanics we come across some peculiar phenomena when the constituent particles move with a high velocity comparable to that of light. The theory of relativity enables one to understand the high energy phenomenon in microscopic as well as macroscopic world.

Frame of Reference

Any physical phenomenon takes place in space and time. It is not possible to investigate any physical phenomenon without introducing a reference frame relative to which all observations will be made. A reference frame may be defined as the system of coordinate axes which is considered to be fixed with respect to which the position of a particle is measured.

There are two types of reference frames – (a) inertial frame of reference and (b) non-inertial frame of reference.

Inertial Frame of Reference

It is such a frame which is either at rest or in motion with uniform velocity. Newton's laws of motion hold good in the inertial frame of reference.

Non-inertial Frame of Reference

It is such a reference frame which moves with acceleration in a straight line or rotates.

Space and Time Fra..e

To determine the exact location and the exact time of occurence of an event, we need time (t) as a coordinate along with three space coordinates (x, y, z).

A frame of reference which has four co-ordinates x, y, z and t in called a space-time reference frame. The space coordinates (x, y, z) represent a 'point' in space and the space coordinates (x, y, z) along with time coordinate (t) represent an event in space and time i.e. the coordinates (x, y, z, t) represent an event.

Inertial Reference System

If we consider a reference system at rest, then this system as well as all other reference systems moving with respect to it with uniform velocities are called inertial reference systems.

Galilean Transformation

It is a method of transformation of an event from one inertial frame to another one. Let us consider two inertial frames of reference represented by S and S'. Let the frame S be at rest and the frame S' be moving with a constant velocity 'v' along the positive x-direction (Fig. 1).

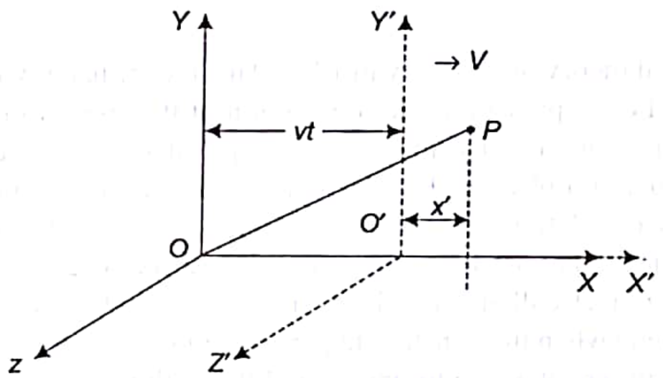


Fig. 1 Two inertial reference frames

Let an event occur at the point P. The coordinates of P with respect to the frames S and S' are (x, y, z, t) and (x', y', z', t') respectively. At the time $t = 0$ the origins of S and S' are coincident at O. As the frame S' is moving with the constant velocity V with respect to S, the origin O' of S' gets shifted from O of S by a distance vt during time t. As there is no motion along Y-and Z-axes, we can write the follow transformation equations (vide Fig. 1):

$$x' = x - vt \quad \text{...(1)}$$

$$y' = y \quad \text{...(2)}$$

$$z' = z \quad \text{...(3)}$$

$$t' = t \quad \text{...(4)}$$

These four equations are called Galilian transformation equations.

The coordinates (x, y, z, t) when expressed in terms of the coordinates (x', y', t', t') give us the following four equations:

$$x = x' + vt \quad \text{...(5)}$$

$$y = y'$$

$$z = z'$$
$$t = t'$$

These four equations are known as inverse galilean transformation equations

Now, differentiating equations (1), (2), (3) and (4) with respect to time, we get

$$\left.\begin{array}{l} \dot{x}' = \dot{x} - v \\ \dot{y}' = \dot{y} \\ \dot{z}' = \dot{z} \end{array}\right\} \Rightarrow \left.\begin{array}{l} v_x' = v_x - V \\ v_y' = v_y \\ v_z' = v_z \end{array}\right\} \quad ...(6)$$

where $\dot{x} = \dfrac{dx}{dt}$ and $\dot{x}' = \dfrac{dx}{dt}$

Equation (6) implies that the veloity of a particle is different in two different systems of reference.

The equations (6) may be used to transform velocity from one reference frame to another one.

Differentiating equation (6), one gets

$$\left.\begin{array}{l} \dot{v}_x' = \dot{v}_x \\ \dot{v}_y' = \dot{v}_y \\ \dot{v}_z' = \dot{v}_z \end{array}\right\} \quad ...(7)$$

and

i.e.,

$$\left.\begin{array}{l} \ddot{x}' = \ddot{x} \\ \ddot{y}' = \ddot{y} \\ \ddot{z}' = \ddot{z} \end{array}\right\} \quad ...(8)$$

and

where $\ddot{x} = \dot{v} = \dfrac{d^2x}{dt^2}$

So, from equation (7) or (8), it is observed that the acceleration of a particle is invariant under the galileon transformation.

Let m be the mass of the particle at the point P and F be the force acting on it in the inertial frame S.

$\therefore$ $F = m\ddot{x}$

Similarly in the inertial frame S', we get

$$F' = m\ddot{x}'$$

$\therefore$ $F = F'$ $\qquad [\because \ \ddot{x} = \ddot{x}']$

So, Newton's laws of motion remain unchanged (or invariant) under the Galilean transformation. So, in general, the laws of mechanics remain same in all inertial frames of reference.

Michelson – Morley Experiment

Michelson and Morley assumed that light waves propagate through a hypothetical medium (called ether) with a speed of $c = 3 \times 10^8$ ms^{-1} which is supposed to be present every where. The earth moves through the ether with a linear speed of $v = 3 \times 10^4$ ms^{-1} approximately. As the earth moves around the sun, we also move through ether with the same speed. If two beams of light are sent out such that one travels in the direction of motion of the earth and the other moves at right angles to that of the earth, Then these two beams would require different times for round-trip journeys through the same distance. If this time–difference could be measured, one could determine the velocity of earth with respect to ether that is one can detect the motion of the earth through ether by performing one optical experiment on the surface of earth it self.

To perform the experiment, a Michelson inter ferometes was used (Fig. 2).

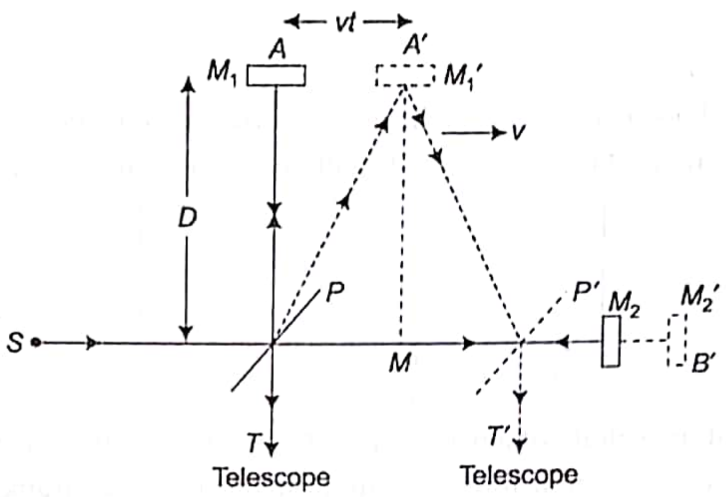


Fig. 2 The Michelson–Morley experiment

A beam of light from the source S is made incident on a semi-silvered glass plate P kept inclined at an angle of 45° with the incident beam. Each incident beam is split up into two beams. One of the two beams is reflected by glass plate P and the other one is transmitted through the glass plate P. The reflected light beam travels towards mirror M_1 and is reflected back by M_1 towards the glass plate P. A part of this beam is then transmitted through glass plate P and enters the telescope T. The other part of the incident beam (which is transmitted) travels towards the mirror M_2 which reflects the beam back towards the glass plate P. A part of this beam is then partially reflected into the telescope T. In the actual arrangement of the interferometer, another glass plate (not shown) is introduced in the path of the transmitted beam in order to make the paths of the two light beams equal in glass. The two beams enter the telescope along a coincident path and interfere with each other. A pattern of interference having bright and dark fringes is obtained.

Let both of the mirrors (M_1 and M_2) be at the same distance D from the glass plate P. If the apparatus remains stationary in ether, there the two beams take the same time to return back to glass plate P and accordingly meet in the same phase at the glass plate as well as telescope. But, the earth along with the apparatus is moving with a velocity V. As can be seen, the incident beam and the apparatus move in the same direction, the incident beam strikes the glass plate when it is in the position P. As the apparatus is in motion, the paths of the reflected and the transmitted beams and their reflections at the mirrors M_1 and M_2 will be as has been shown by the dotted lines. Obviously the time taken by the two waves for their 'round trip' journeys through the same distance will not be equal.

The transmitted beam travels towards the mirror M_2 with relative velocity $c - v$. And having got reflected at M_2, it travels towards the glass plate P with a velocity $c + v$. Hence, the required time for its round trip is given by

$$t_1 = \frac{D}{c-v} + \frac{D}{c+v} = \frac{2Dc}{c^2 - v^2}$$

or

$$t_1 \approx \frac{2D}{c}\left(1 + \frac{v^2}{c^2}\right) \qquad \text{...(9)}$$

The time taken by the [$\therefore \quad v << c$] reflected beam for its round–trip is given by

$$t_2 = 2t' \qquad \text{...(10)}$$

When t' is the time required by the beam to travel from P to A'.

Now, from the triangle PA′ M, one gets

$$c^2 t'^2 = v^2 t'^2 + D^2$$

or,
$$t'^2 = \frac{D^2}{c^2 - v^2}$$

or,
$$t' = \frac{D}{(c^2 - v^2)^{\frac{1}{2}}}$$

or,
$$t' \approx \frac{D}{C}\left(1 + \frac{v^2}{2c^2}\right) \qquad ...(11)$$

Hence
$$t_2 = 2t' = \frac{2D}{c}\left(1 + \frac{v^2}{2c^2}\right) \qquad ...(12)$$

From equations (9) and (12), we get the difference in the times required by the two beams for their round trips is given by

$$t_1 - t_2 = \frac{Dv^2}{c^3} \qquad ...(13)$$

A time difference of one period (T) would amount a path difference of λ (one wavelength), which will in turn, amount to a displacement of one fringe across a particular point (i.e., the cross-point of the cross wires) in the field of view of the telescope. Hence, a time difference $(t_1 - t_2) = \frac{Dv^2}{c^3}$ causes $\frac{Dv^2}{c^3 T} = \frac{Dv^2}{c^2 \lambda}$ fringes to be displaced across the cross-point of the crosswires. Here the relation $CT = \lambda$ has been used. The displacement of the fringes, in this case, cannot be seen since the apparatus is at rest with respect to the observer all the time. Hence, no information regarding the time difference between the two paths can be obtained with an apparatus which is stationary on the earth. For this reason, the apparatus is then slowly turned through 90° so that the two light beams interchange their paths. This is equivalent to introduction of an additional path difference which would make the displacement of the fringes noticeable. Because of this act of rotation $\frac{2Dv^2}{c^2 \lambda}$ fringes should be displaced.

In the Michelson–Morley experiment effective distance was 11 m i.e., $D = 11$ m. The wavelenght of light $\lambda = 5.9 \times 10^{-7}$ m, $v = 3 \times 10^4$ ms^{-1} and $C = 3 \times 10^8$ ms^{-1}.

So, $\frac{2Dv^2}{c^2 \lambda} = 0.37$. Thus, 0.37 fringe was expected to be displaced across the cross-point of the cross–wires. Their apparatus was capable of noticing even 0.01 fringe i.e., one hundredth of a fringe. The observed displacement was extremely small as compared to the theoretical value of 0.37 fringe–width. So, the result was not consistent. The experiment was performed in different reasons of the year and at different places. But, the result was the same. Thus, the motion of the earth with respect to ether cannot be detected.

The negative result led them to the following conclusions: (i) The hypothesis of existence of ether was rendered untenable by demonstrating that ether has no measurable properties (ii) The speed of light in free space is constant irrespective of the motion of the source or the observer.

Einstein's Special Theory of Relativity

Due to invariance of the velocity of light in all inertial reference frames as has been seen above, Einstein put forward his special theory of relativity in 1905. The special theory of relativity is based on the following two postulates:

(i) All the basic laws of physics remain same in all inertial frames of reference.

(ii) The speed of light in vacuum is same in all inertial frames of reference irrespective of the motion of the source relative to the observer.

The consequence of the first postulate is that there is no universal (or absolute) reference frame. The second postulate is the consequence of the experiment of Michelson and Morley.

Lorentz Transformation

A set of transformation equations which satisfy the postulates of the special theory of regativity is known as the set of Lorentz transformation equations. This set provides us with a method of obtaining the co-ordinates of a point in a certain reference frame relative to another inertial reference frame and vice-versa.

The Galilean transformation equations are

$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\} \quad \text{...(14)}$$

and

And the velocity transformation equation are given by

$$\left.\begin{aligned} u_{x'} &= u_x - v \\ u_{y'} &= u_y \\ u_{z'} &= u_z \end{aligned}\right\} \quad \text{...(15)}$$

From equation (15) we can get the idea that these equations violate the first postulate of the special theory of relativity. If we consider the velocity of light in the two inertial frames S and S' to be c and c' respectively, then from equation (15), we get

$$c' = c - v \quad \text{...(16)}$$

Here, we have considered different velocities of light in different reference frames this is not acceptable in case of the special theory of relativity. So, in order to satisfy the postulates of the special theory of relativity, we must get a set of better transformation equations than the galileon transformation equations which are termed as Lorentz transformation equations as mentioned above.

In order to develop these equations, let a possible relation between x and x' be of the form

$$x' = k\,(x - vt) \quad \text{...(17)}$$

Where k is a constant of proportionality. The constant k does not depend on x and t but it may be a function of V. The relation (17) is linear in x and x' and it can be easily reduced to the form $x' = x - vt$ which is correct in newtonian mechanics if $k = 1$ when $v << c$. The inverse relation of equation (17) can be written as

$$x = k'\,(x' + vt') \quad \text{...(18)}$$

[Note the interchange of primed and unprimed variables]

In this new transformation we are not taking $t = t'$ relation. That is in Lorentz transformation $t' \neq t$ and v has been replaced by $-v$. The other relations are as in case of galileon transformation, i.e.,

$$\left.\begin{aligned} y' &= y \\ \text{and} \quad z' &= z \end{aligned}\right\} \qquad \text{...(19)}$$

From equations (17) and (18), we get

$$x = k'(x' + vt')$$

or, $$x = k'[k(x - vt) + vt']$$

or, $$x = kk'(x - vt) + k'vt'$$

or, $$k'vt' = -kk'(x - vt) + x$$

or, $$k'vk' = kk'vt - kk'x + x$$

or, $$t' = kt + (-kk' + 1)\frac{x}{vk}$$

or, $$t' = kt + (1 - kk'/k'v)\,x \qquad \text{...(20)}$$

In order to determine the value of k and k', we use the second postulate. Let at $t = 0$, the origin of the two systems S and S' coincide with each other. Let this instant correspond to $t' = 0$. Suppose that a light signal is given out from the common origin of the frames S and S' at $t = t' = 0$. The signal propagates in the two systems satisfying the equations

$$x = ct \qquad \text{...(21)}$$

and $$x' = ct' \qquad \text{...(22)}$$

Let us replace x' and t' in equation (22) by using equations (17) and (20),

$\therefore$ $$k(x - vt) = ckt + \left(\frac{1 - kk'}{k'v}\right)cx$$

Hence, $$x = ct\left[\frac{(1 + v/c)}{1 - \left(\frac{1}{kk'} - 1\right)\frac{c}{v}}\right] \qquad \text{...(23)}$$

Now, comparing equations (21) and (23) we can write,

$$1 = \frac{(1 + v/c)}{1 - \left(\frac{1}{kk'} - 1\right)\frac{c}{v}}$$

or, $$1 - \left(\frac{1}{kk'} - 1\right)\frac{c}{v} = 1 + \frac{v}{c}$$

or, $$\left(1 - \frac{1}{kk'}\right)\frac{c}{v} = \frac{v}{c}$$

or, $$1 - \frac{1}{kk'} = \frac{v^2}{c^2}$$

or $$kk' = 1/\left(1 - \frac{v^2}{c^2}\right)$$

or, $$\sqrt{kk'} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{...(24)}$$

As can be seen in equation (24), kk' depends on v^2 (and not on v), the relative velocity of S' with respect to S.

In fact, we cannot choose between the two reference frames S and S' without taking account of the sign of v. But the sign of v does not affect the dependence of kk' on v. So, we choose

$$k = k' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad ...(25)$$

Substituting there values of k and k' in equations (17) and (20) and including equations (19), we get the following set of equations:

$$\left.\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t' - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned}\right\} \qquad ...(26)$$

The set of equations (26) is known as Lorentz transformation equations. The inverse Lorentz transformation can now be written as

$$\left.\begin{aligned} x &= \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \\ y &= y' \\ z &= z' \\ t &= \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned}\right\} \qquad ...(27)$$

Thus, the measurement of position and time are found to depend upon the frame of reference of the observer concerned. It $v << c$, we get the galileon transformation from Lorentz transformation.

Addition of Velocities

Let us consider a body which is moving with respect to both the frames S and S'. An observer in the frame S measures the Components of Velocity $\overline{V}$ of the body as

$$V_x = \frac{dx}{dt}, \quad V_y = \frac{dy}{dt} \quad \text{and} \quad V_z = \frac{dz}{dt} \qquad ...(28)$$

And an observer in the frame S' maker the following measurements of the velocity $\overline{V}'$ of the body as

$$V'_x = \frac{dx'}{dt'}, \quad V'_y = \frac{dy'}{dt'} \quad \text{and} \quad V'_z = \frac{dz'}{dt'} \qquad ...(29)$$

Now, from Lorentz transformations, we get

$$\left.\begin{aligned} dx' &= \frac{dx - vdt}{\sqrt{1-\frac{v^2}{c^2}}} \\ dy' &= dy \\ dz' &= dz \\ dt' &= \frac{dt - \frac{vdx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \end{aligned}\right\} \quad \text{...(30)}$$

$$\therefore \quad V'_x = \frac{dx'}{dt'} = \frac{dx - vdt}{dt - \frac{vdx}{c^2}}$$

or

$$V'_x = \frac{V_x - v}{1 - \frac{v}{c^2}V_x} \quad \text{...(31)}$$

The y – component of velocity is

$$V'_y = \frac{dy'}{dt'} = \frac{dy}{\frac{\left(dt - \frac{v}{c^2}dx\right)}{\sqrt{1-\frac{v^2}{c^2}}}}$$

or

$$V'_y = \frac{V_y\sqrt{1-\frac{v^2}{c^2}}}{1 - \frac{V_x v}{c^2}} \quad \text{...(32)}$$

Similarly

$$V'_z = \frac{V_z\sqrt{1-\frac{v^2}{c^2}}}{1 - \frac{V_x v}{c^2}} \quad \text{...(33)}$$

The equations (31), (32) and (33) constitute transformation equations for velocity.

If the velocity of the frame S' with respect to the frame S is V and $V << c$, then from equations (31), (32) and (33), we get

$$\left.\begin{aligned} V'_x &= V_x - v \\ V'_y &= V_y \\ V'_z &= V_z \end{aligned}\right\} \quad \text{...(34)}$$

Which are galileon transformation equation for velocities.

The inverse transformations equations for velocities are as follows:

$$V_x = \frac{V'_x + v}{1 + \frac{vV_x}{c^2}}$$

$$V_y = \frac{V'_y\sqrt{1 - v^2\backslash c^2}}{1 + \frac{vV'_x}{c^2}} \qquad \text{...(35)}$$

and
$$V_z = \frac{V'_z\sqrt{1 - v^2/c^2}}{1 + \frac{vV_x}{c^2}}$$

Suppose that a ray of light is emitted in the frame S' (which is moving with a velocity v with respect to the frame S) in the same direction as that of its own motion with respect to S, then $V'_x = c$. The observer in the frame S would measure the velocity of the emitted light ray as

$$V_x = \frac{V'_x + v}{1 + \frac{vV'_x}{c^2}}$$

or,
$$V_x = \frac{c + v}{1 + \frac{vc}{c^2}} = \frac{c + v}{1 + \frac{v}{c}} = \frac{c(c + v)}{(c + v)}$$

$\therefore$
$$V_x = c$$

If velocity of light is c in the frame S', then it is also c in the frame S. This is consistent with the second postulate of the special theory of relativity.

Variation of Mass with Velocity

In classical mechanics mass is considered to be constant. But in relativistic mechanics it can vary with respect to time. So, logically we can assume that mass will be a function of velocity.

$\therefore$ $m = m(u)$ where u is the velocity with which the mass is moving. If $u = 0$, then $m = m_0$, the rest mass.

To determine the dependence of mass on velocity, we shall consider a collision of two bodies and further assume that the law of conservation of momentum and the law of conservation of relativistic masses of the particles are valid.

Let us consider two bodies each of mass m' moving in opposite directions along the x'–axis with velocities u' and $-u'$ as observed from S' (Fig. 3).

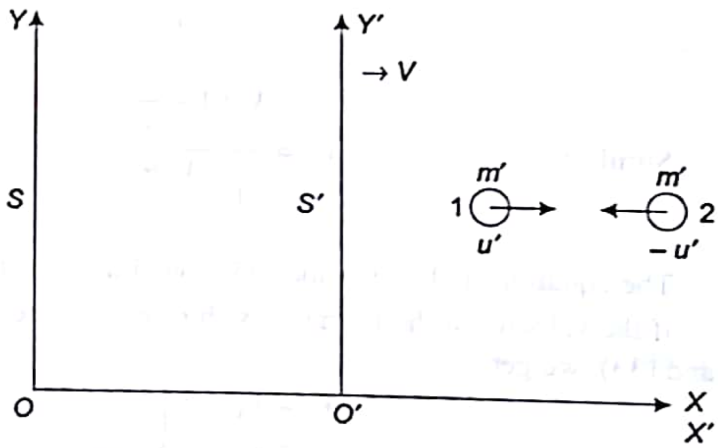


Fig. 3 **Variation of mass with Velocity.**

Let there two bodies collide and coalesce into one single body.

The body thus formed will be at rest according to the law of conservation of momentum in the frame S'.

If the collision is observed from the frame S, the Velocities of the two bodies will be given by

$$u_1 = \frac{u' + v}{1 + \frac{u'v}{c^2}} \quad \text{...(36)}$$

and

$$u_2 = \frac{-u' + v}{1 - \frac{u'v}{c^2}} \quad \text{...(37)}$$

Let m_1 and m_2 be the masses of the two bodies with respect to S.

The mass of the combined body is

$$m = m_1 + m_2 \quad \text{...(38)}$$

The body of mass m moves with a velocity v along the positive x-direction with respect to the frame S as it is at rest in S'.

From the law of conservation of momentum

We can write,

$$m_1 u_1 + m_2 u_2 = mv \quad \text{...(39)}$$

or,

$$m_1\left[\frac{u' + v}{1 + \frac{u'v}{c^2}}\right] + m_2\left[\frac{-u'v}{1 - \frac{u'v}{c^2}}\right] = (m_1 + m_2)\, v$$

or,

$$\frac{m_1}{m_2}\left[\frac{u' + v}{1 + \frac{u'v}{c^2}}\right] + \left[\frac{-u' + v}{1 - \frac{u'v}{c^2}}\right] = \left(\frac{m_1}{m_2} + 1\right) v$$

or,

$$\frac{m_1}{m_2}\left[\frac{u' + v}{1 + \frac{u'v}{c^2}}\right] - \frac{m_1}{m_2} v = v + \frac{u' - v}{1 - \frac{u'v}{c^2}}$$

or,

$$\frac{m_1}{m_2}\left[\frac{u'\left(1 - \frac{v^2}{c^2}\right)}{1 + \frac{u'v}{c^2}}\right] = \frac{u'\left(1 - \frac{v^2}{c^2}\right)}{1 - \frac{u'v}{c^2}}$$

or,

$$\frac{m_1}{m_2} = \frac{1 + u'v/c^2}{1 - u'v/c^2} \quad \text{...(41)}$$

Now, divinding both sides of equation (36) by c we get

$$\frac{u_1}{c} = \frac{(u' + v)}{c\left(1 + \frac{u'v}{c^2}\right)}$$

Scanned by CamScanner

or, $$\frac{u_1^2}{c^2} = \frac{(u'+v)^2}{c^2\left(1+\frac{u'v}{c^2}\right)^2}$$

or, $$1-\frac{u_1^2}{c^2} = 1-\frac{(u'+v)^2}{c^2\left(1+\frac{u'v}{c^2}\right)^2}$$

or, $$1-\frac{u_1^2}{c^2} = \frac{c^2\left(1+\frac{u'v}{c^2}\right)^2-(u'+v)^2}{c^2\left(1+\frac{u'v}{c^2}\right)^2}$$

Hence $$1-\frac{u_1^2}{c^2} = \frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{\left(1+\frac{u'v}{c^2}\right)^2}$$

or, $$\sqrt{1-\frac{u_1^2}{c^2}} = \sqrt{\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{\left(1+\frac{u'v}{c^2}\right)}}$$

or, $$\left(1+\frac{u'v}{c^2}\right) = \sqrt{\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{\left(1-\frac{u_1^2}{c^2}\right)}} \quad ...(42)$$

Similarly from equation (37), we get

$$\left(1-\frac{u'v}{c^2}\right) = \sqrt{\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{\left(1-\frac{u_2^2}{c^2}\right)}} \quad ...(43)$$

Now, substituting these values in equation (41), we get

$$\frac{m_1}{m_2} = \frac{\sqrt{1-\frac{u_2^2}{c^2}}}{\sqrt{1-\frac{u_1^2}{c^2}}} \quad ...(44)$$

If the velocity of the mass m_2 with respect to the frame S be zero, i.e., $u_2 = 0$, then its mass m_2 can be denoted by m_o. The symbol m_o gives the (rest) mass of the second body when it is at rest with respect to the frame S.

Let $u_1 = v$, i.e., the velocity of the first body with respect to the frame S is v.

So, the equation (44) can be written as follows (by denoting m_1 by m):

$$\frac{m}{m_o} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\therefore \quad m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad ...(45)$$

This gives the variation of mass with respect to velocity.

The momentum is defined as

$$mv = \frac{m_o}{\sqrt{1 - v^2/c^2}} v$$

The momentum is conserved in the theory of relativity also.

In relativistic form Newton's second law of motion is written as

$$F = \frac{d}{dt}(mv) = \frac{d}{dt}\left(\frac{m_o v}{1 - \frac{v^2}{c^2}}\right)$$

Mass and Energy Relation

The kinetic energy T of a moving body is the work done in giving it that state of motion starting from rest.

So, the kinetic energy T' is given by

$$T = \int_0^t F dr$$

where F is the component of force parallel to dr.

Now, we can express force as

$$F = \frac{d}{dt}(mv)$$

$$\therefore \quad T = \int_0^t \frac{d(mv)}{dt} dr$$

$$\text{or,} \quad T = \int_0^{mv} v d(mv) \qquad \left[\because \frac{dr}{dt} = v\right]$$

$$\text{or,} \quad T = \int_0^v v d \frac{m_o v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

or, $$T' = \frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_o \int_0^v \frac{v\,dv}{\sqrt{1-\frac{v^2}{c^2}}}$$ **[Integrating by parts]**

or, $$T' = \frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_o c^2 \left(\sqrt{1-\frac{v^2}{c^2}} - 1\right)$$

or, $$T = \frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_o c^2 \sqrt{1-\frac{v^2}{c^2}} - m_o c^2$$

or, $$T = \frac{m_o v^2 + m_o c^2 - m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} m_o c^2$$

or, $$T = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_o c^2$$

or, $$T = mc^2 - m_o c^2 \qquad ...(46)$$

or, $$T = \Delta mc^2 \quad \text{where} \quad \Delta m = m - m_o$$

So, the kinetic energy of a body is the increase in mass multiplied by the square of the speed of light in free space.

The equation (46) can be written as

$$mc^2 = T + m_o c^2$$

i.e., total energy of a body is equal to sum of its kinetic energy and rest mass energy.

If we represent total energy by E and the rest mass energy by E_o, then equation (46) can be written as

$$E = E_o + T \qquad ...(47)$$

where $E = mc^2$ and $E_o = m_o c^2$

The equation $E = mc^2$ expresses the universal equivalence of mass and energy.

It is also called Einstein's mass energy relation.

One can easily convert the kinetic energy T to its classical value $1/2\ mv^2$ as follows:

$$T = mc^2 - m_o c^2$$

or, $$T = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_o c^2$$

or, $$T' = m_o c^2 \left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right]$$

or, $$T = m_o c^2 \left[\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} - 1 \right]$$

or, $$T = m_o c^2 \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots - 1\right)$$ [Neglecting higher terms as classically $v << c$]

or, $$T = \frac{1}{2} m_o v^2 \qquad \ldots(48)$$

Energy–Momentum Relation

As stated earlier, the relativistic momentum (p) of a moving particle of mass m is given by

$$p = mv = \frac{m_o v}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \ldots(49)$$

The total relativistic energy is given by

$$E = mc^2 = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} c^2 \qquad \ldots(50)$$

So, by using equations (49) and (50), we can write

$$E^2 - p^2 c^2 = \frac{m_o^2 c^4}{\left(1 - \frac{v^2}{c^2}\right)} - \frac{m_o^2 v^2 c^2}{\left(1 - \frac{v^2}{c^2}\right)}$$

or, $$E^2 - p^2 c^2 = m_o^2 c^4 \left\{ \frac{1 - v^2/c^2}{\left(1 - \frac{v^2}{c^2}\right)} \right\}$$

or, $$E^2 - p^2 c^2 = m_o^2 c^4$$

or, $$E^2 = m_o^2 c^4 + P^2 c^2$$

$\therefore$ $$E = \sqrt{m_o^2 c^4 + p^2 c^2} \qquad \ldots(51)$$

The equation (51) gives the relation between the total relativistic energy (E) and the momentum (P) of the body.

The equation $E^2 - P^2 c^2 = m_o^2 c^4$ is an invariant.

If the rest mass of a particle is zero, then equation (51) reduces to

$$E = pc$$

APPENDIX

D

Experiments

EXPERIMENT NO. 1

Aim To determine wavelength of sodium light having formed Newton's rings.

Apparatus Travelling microscope, support for glass-plate inclined at an angle of 45°, plano-convex lens, thin glass plate, source of sodium light and spherometer.

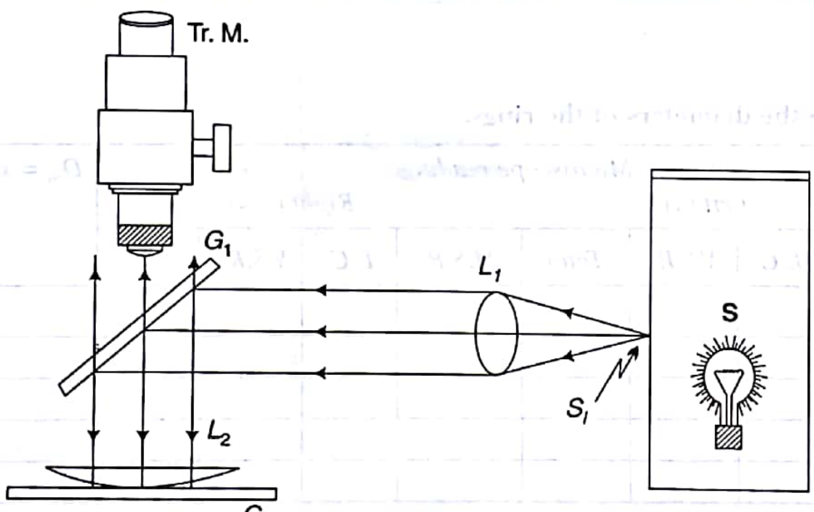


Fig. D.1a Newton's ring experimental set up. L_1: Biconvex lens, L_2: Planoconvex lens, G_1: Glass plate inclined at 45°, G_2: Horizontal glass plate, S: Sodium lamp, S_l: slit, *Tr. M.*: Travelling Microscope

Working Formula

$$\lambda = \frac{D_{n+m}^2 - D_n^2}{4Rm}$$

where D_{n+m} = diameter of $(n+m)^{th}$ ring.

D_n = diameter of n^{th} ring.

and R = radius of curvature of the convex surface of the plano-convex lens.

Procedure

1. Place the sodium vapour lamp in such a way that its light-passing slit remains at the focus of a vertical biconvex lens. The light rays coming out of the lens will be parallel.
2. The inclined glass plate (inclined at 45°) will reflect the light beam vertically downward.
3. The plano-convex lens is placed under the inclined glass plate on another horizontal glass plate so that its plane surface remains on top and the spherical surface lies on the horizontal glass plate. There will be air film between the lens and the glass plate.
4. Light rays reflected from the spherical surface of the plano-convex lens and the horizontal glass plate are received through the travelling microscope.
5. Circular rings are formed in the thin air film between the plano-convex lens and the horizontal glass plate.
6. Adjust the microscope to see the bright (and dark) rings distinctly.
7. Make one of the cross-wires (vertical one) tangent to a bright ring (say 10th) at the right side. Take the reading from the scale attached to the microscope. Then move the cross-wire towards left and make it tangent at the right of the 7^{th} ring and take the reading. Similarly, take reading on the right side of the fourth ring.
8. Moving the cross-wire in the same direction take readings on left sides of the 4^{th}, 7^{th} and 10^{th} rings.
9. Measure the radius of curvature (R) of the curved surface of the plano-convex lens using spherometer (usually it is given).
10. The observation for diameters (D) of the rings are tabulated as in the Table D.1(given below):

Observations

Table D.1 To measure the diameters of the rings.

No. of rings	*Microscope reading*								$D_m = x \sim y\ (cm)$	$D_m^2\ (cm^2)$
	Left (x)				*Right (y)*					
	M.S.R	*L.C.*	*V.S.R.*	*Total*	*M.S.R.*	*L.C.*	*V.S.R.*	*Total*		

Calculations

$$\lambda = \frac{D_{n+m}^2 - D_n^2}{4mR}\ cm^2$$

Plot the graph between D_n^2 and the number of rings (n). Measure the slope (s) of the line which is given

$$s = \frac{\Delta(D_n^2)}{\Delta n}$$

∴ wave length, $$\lambda = \frac{D_{n+m}^2 - D_n^2}{4mR} = \frac{\Delta(D_n^2)}{4\Delta nR}$$

i.e.,
$$\lambda = \frac{s}{4R}$$

Results

Percentage Error

Discussions

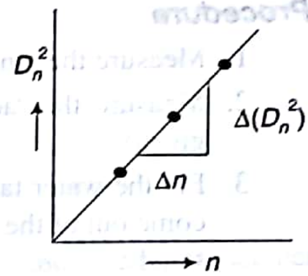


Fig. D.1b

EXPERIMENT NO. 2

Aim Determination of the coefficient of viscosity of water by Poiseulle's Capillary flow method.

Apparatus Constant-head apparatus, capillary tube, beaker, measuring cylinder, stop-watch, rubber tube for connection, metre scale, pinch cock and spirit level.

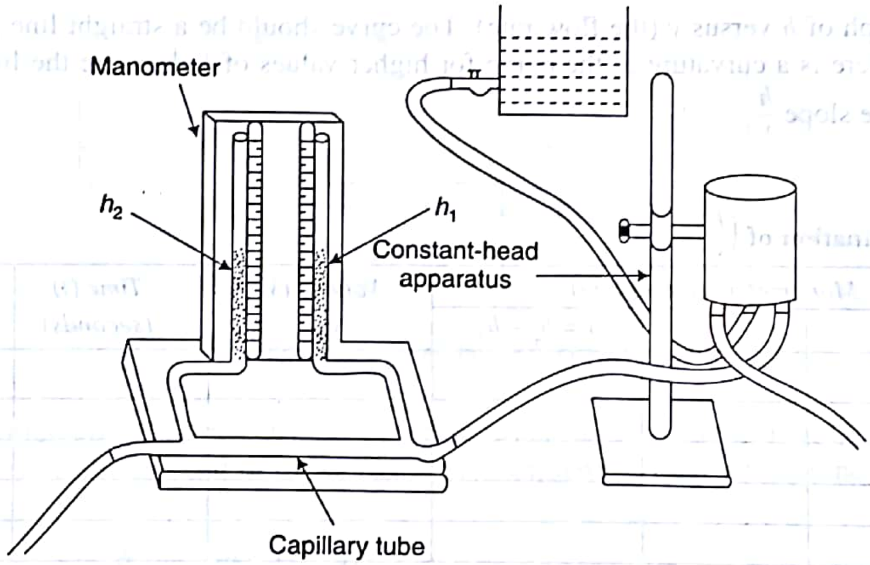


Fig. D.2 Experimental set up for viscosity measurement

Working Formula

$$\eta = \frac{\pi \rho g r^4}{8l}\left(\frac{h}{v}\right)$$

where η = coefficient of viscosity,
ρ = density of the liquid,
g = acceleration due to gravity,
r = radius of the capillary tube,

Scanned by CamScanner

l = length of the capillary tube,

$h = (h_2 - h_1)$ = pressure head,

and v = rate of flow of the liquid (in volume).

Procedure

1. Measure the length of the capillary tube by using a metre scale.
2. Measure the radius of the sample capillary tube by using a travelling microscope (usually it is given).
3. Fix the water tank of the viscosity apparatus at a suitable height. Allow the water from the tank to come out of the tube slowly. Wait until the levels of water at the two arms of the manometer become steady.
4. Let water flow through the rubber tube slowly by making use of the pinch cock. The water level in the nearer arm of the (rubber tube) opening will come down. Fix it at such a level that water flow through the capillary tube remains laminar.
5. Collect water from the output end of the rubber tube in a measuring cylinder for five minutes. Then measure the volume of the collected water and from this calculate the flow rate of water (in volume/s). Repeat this step three more times having kept the water level different from the first case in the nearer arm of the manometer.
6. Draw a graph of h versus v (the flow rate). The curve should be a straight line passing through the origin. If there is a curvature in the curve for higher values of h then use the linear portion of it to calculate the slope $\frac{h}{v}$.

Observations

Table D.2 Determination of $\left(\frac{h}{v}\right)$

No. of obs.	*Manometer reading (cm)*			*Volume (V)* cm^3	*Time (t)* *(seconds)*	*Flow rate* $v = V/t$
	h_1	h_2	$h = h_2 - h_1$			

Use of Graph

$$\text{The slope} = \frac{\Delta h}{\Delta v}$$

Calculations

Results

Percentage of Error

Discussions

EXPERIMENT NO. 3

Aim To determine the value of unknown resistance by using Carey-Foster's bridge and also resistance per unit length of the bridge wire.

Apparatus Carey Foster's bridge, two equal one ohm resistances, power supply unknown resistance, table galvanometer, resistance box, plug commutator.

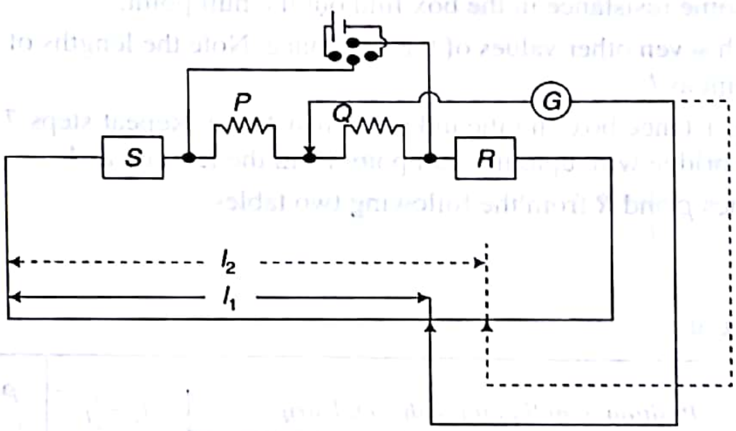


Fig. D.3 Carey Foster's bridge

Theory Carey Foster's bridge connection has been shown in the diagram of Fig. D.3. If one gets the null point at a length l_1 from the left end of the bridge wire, then having used wheatstone bridge principle one gets,

$$\frac{P}{Q} = \frac{S + \alpha + l_1 \rho}{R + \beta + (100 - l_1)\rho} \quad ...(1)$$

where α and β are end corrections and ρ is the resistance per unit length of the bridge wire.

If one interchanges the positions of S and R and gets a null point at a length l_2 from the left end, then

$$\frac{P}{Q} = \frac{R + \alpha + l_2 \rho}{S + \beta + (100 - l_2)\rho} \quad ...(2)$$

Solving the two equations (1) and (2), one can get

$$\rho = \frac{S - R}{l_2 - l_1} \quad ...(3)$$

or,
$$R = S - \rho\,(l_2 - l_1) \quad ...(4)$$

Now, if one replaces R by a thick metal strip, then $R = 0$ and Eq. (3) reduces to

$$\rho = \frac{S}{(l_2 - l_1)} \quad ...(5)$$

Procedure

1. Connect the circuit of Carey Foster's bridge as shown in the diagram of Fig. D.3.
2. Put a fractional resistance box in the extreme left gap and a metal strip in the extreme right gap.
3. Put two equal resistances P and Q (each of 1 ohm resistance) in the two middle gaps.
4. Put a small resistance (say, 0.5 ohm) in the box. Measure the null point.
5. Repeat Step 4 seven more times with different values of small resistance in the resistance box. Note these values as l_1.
6. Interchange the resistance box and the metal strip. Repeat steps 4 and 5 with same values of the resistance. Note these values as l_2.
7. Put the resistance box in the left gap and an unknown resistance (which is to be measured) in the right gap. Having put some resistance in the box find out the null point.
8. Repeat Step 7 with seven other values of the resistance. Note the lengths of the bridge wire from left end to the null point as l_1.
9. Interchange the resistance box and the unknown resistance. Repeat steps 7 and 8 seven times. Note the lengths of the bridge wire upto the null point from the left end as l_2.
10. Calculate the values ρ and R from the following two tables:

Observations

Table D.3a Measurement of ρ

No. of Obs.	*S in ohm*	*Position of null point with metal strip at*		$l_2 - l_1$	$\rho = \frac{S}{l_2 - l_1}$	*Mean ρ*
		right gap l_1(cm)	*left gap l_2(cm)*	*(cm)*	*ohm/cm*	*ohm/cm*

Table D.3b Measurement of unknown resistance (R)

No. of Obs.	*S in ohm*	*Position of null point when the unknown resistance is*		$l_2 - l_1$	$R = S - \rho(l_2 - l_1)$	*Mean R*
		right gap l_1(cm)	*left gap l_2(cm)*	*(cm)*	*(ohm)*	*(ohm)*

Results

Percentage Error

Discussions

EXPERIMENT NO. 4

Aim Determination of numarical aperture and the energy loss related to optical fibre.

Apparatus Fibre optic analogue transmitter and receiver kits, one-and five-metre PMMA fibre cords, inline SMA adaptors.

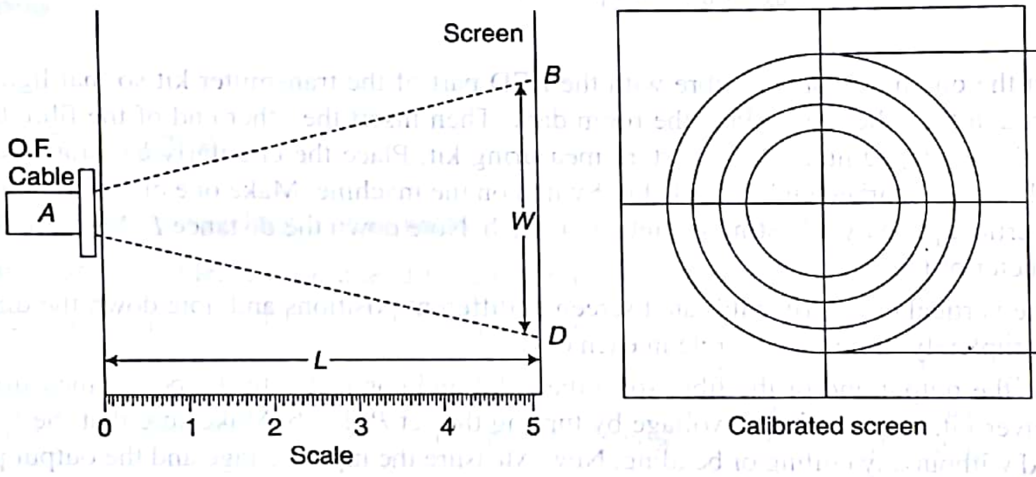


Fig. D.4 Measurement of numerical aperture

Theory The numerical aperture (N_a) of an optical fibre is the amount of light what is collected by it at its end.

The numerical aperture $N_a = \mu \sin \theta$

For air medium $\mu = 1$

so, $N_a = \sin \theta$

In the Fig. D.4, let the light beam from the fibre end A fall on the screen $BD = W$.

$$\therefore \quad N_a = \frac{W}{(4L^2 + W^2)^{1/2}}$$

where L is the distance between the end A of the optical fibre and the screen BD. W is the diameter of the illuminated circle in the screen BD.

The loss of optical energy is a function of the length of the optical fibre. The loss is expressed in the unit of decibel per unit length. Losses also occur in the optical fibre when one joins two fibres by inline adaptor. In this experiment loss per unit length as well as loss due to fibre to fibre joints will be calculated.

If P_0 be the power of the light energy entering an optical fibre of length L and P be the power output then loss due to the fibre is measured as $-10\log_{10}\left(\frac{P}{P_0}\right)$ in dB. In case of a pure optical fibre loss in equal to $\in L$.

If A' be the loss due to the adaptors, then the total loss is equal to $\in L + A'$. If P_{01} and P_{02} are the power losses due to fibres of lengths L_1 and L_2 and P_{03} is that due to the combination when the fibres are joined with an inline SMA of loss contribution A, then the loss due to fibre of one metre length and the SMA is given by

$$P_{03} - P_{02}$$

and similarly loss due to the fibre of length 5 metre and the SMA is given by

$$P_{03} - P_{01}$$

$\therefore$ Loss per unit length, $\in = \dfrac{P_{02} - P_{01}}{L_2 - L_1}$ dB/m

And the loss due to the inline SMA (fibre-to-fibre joint) is given by

$$A = (P_{03} - P_{02}) - \in L_1,\ \text{dB/m}$$

Procedure

1. Connect the one metre optical fibre with the LED part of the transmitter kit so that light is seen to pass through the other end. Make the room dark. Then insert the other end of the fibre through the hole of the L-shaped numerical aperture measuring kit. Place the circularly calibrated screen plate vertically on the marked end of the L-kit. Switch on the machine. Make one circle completely bright on the vertical plate by adjusting the intensity knob. Note down the distance L. Measure the value of the diameter of the circle (W).
2. Move the vertical circularly calibrated screen at different positions and note down the diameter (W) of the completely illuminated circle in each case.
3. Take out the output end of the fibre from the L-kit and connect it to the power measuring port of the receiver kit. Apply an input voltage by turning the set P_0 knob. Make sure that the fibre is fully stretched without any coiling or bending. Now, Measure the input voltage and the output power with the help of a multimeter. This loss is P_{01}.
4. Repeat Step 3 for the five metre fibre. The input voltage should be same as earlier. This power loss is P_{02}.
5. Repeat Step 3 by joining the two fibres by inline SMA. This power loss in P_{03}.

Observations

Table D.4a Determination of numerical aperture

Sl. No	*L(mm)*	*W(mm)*	*Na = W/(4L² + W²)*	*Mean N_a*

Acceptance angle = $\sin^{-1}(N_a)$ =

Table D.4b Calculation of loss of power

Sl. No.	P_{01}	P_{02}	$\in = \dfrac{P_{02}-P_{01}}{L_2-L_1}$	$P_{03}^{A_0}$	$P_{03}^{A_1}$	$P_{03}^{A_3}$	$P_{03}^{A_5}$	$A = (P_{03}-P_{02}) - \in L_1$			
								A_0	A_1	A_3	A_5

Results

Percentage Error

Discussions

EXPERIMENT NO. 5

Aim To determine the wavelength of an unknown light by laser diffraction method.

Apparatus Optical table, laser source, diffraction grating, modified spectrometer.

Theory If a parallel beam of light of wavelength λ be incident on a diffraction grating, the light beam gets diffracted and forms a diffracted pattern in the space. This pattern consists of the zeroth order principal maximum and a number of secondary minima and maxima at various orders. The intensity of the maxima decreases with increasing the order number. The pattern is shown in the following diagram.

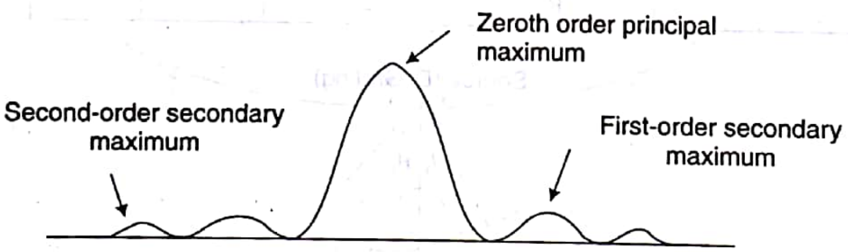


Fig. D.5(a) The diffraction pattern due to a diffraction grating

If θ_n be the angle of diffraction of nth order secondary maximum, then

$$d \sin \theta_n = n\lambda$$

where d is known as grating element. If N be the number of rulings per unit length, then

$$N = \frac{1}{d}$$

$$\therefore \quad \lambda = \frac{d \sin \theta_n}{n} \qquad \text{...(1)}$$

If the value of d is known, then one can find out the wavelength λ of the unknown light.

Procedure

1. Place the grating vertically on the prism table of the modified spectrometer. Switch on the laser source so that laser beam is incident on the grating. Due to diffraction by the grating a number of diffracted beams are seen in the space between the grating and laser detector. These beams can be received by the detector.
2. If one bright beam is received by the detector, then it converts the light signal to equivalent electric signal which is seen on the digital meter.
3. At first by removing the diffraction grating from the prism table, the main light beam is detected. This gives the position of the zeroth-order principal maximum. Then the grating is again placed on the prism table and consequently many bright spots are seen in the space due to diffraction.
4. By moving the detector towards right the second order secondary maximum is received and its position is noted down from the circular scale of the spectrometer. Then the detector is slowly moved towards left and positions of the first-order secondary maximum (right), the central maximum, first order secondary maximum (left) and second-order secondary maximum (left) are detected and noted down as shown in Table D.5:

Observations

Table D.5a Estimation of positions of maxima

No. of Obs.	*Intensity (Digital meter reading)*	*Side Left/Right*	*Order No.*	*Angular position (ϕ)*			
				MSR	*VSR*	*L.C.*	*Total*
1		L	2				
2		L	1				
3		C	0				
4		R	1				
5		R	2				

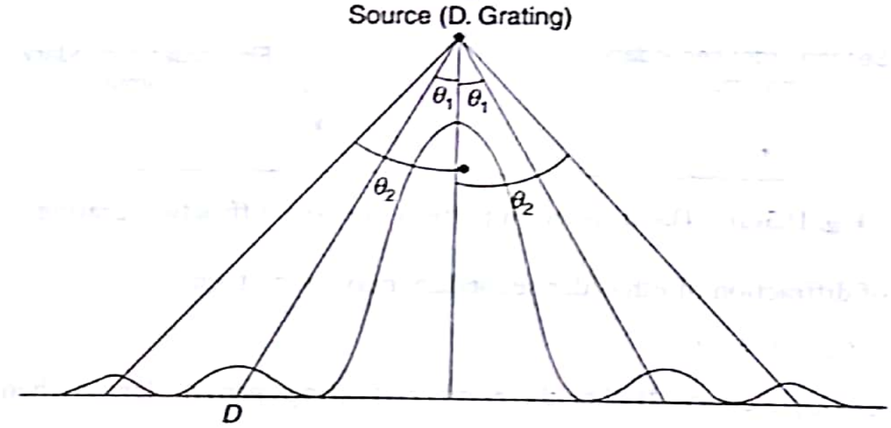


Fig. D.5b Positions of central, first and second order maxima

Table D.5b Measurement of wave length Angular position of central maximum, ϕ_c =

Sl. No.	Side Left/ Right	Order Number	Angular position ϕ	Angle of diffraction (θ)	$\sin\theta$	$\lambda = \frac{d \sin\theta}{n}$	Mean λ
–	–	–	–	$\theta = \phi \sim \phi_c$	–	nm	nm
1	L	2					
2	L	1					
3	R	1					
4	R	2					

Results

Percentage of Error

Discussions

EXPERIMENT NO. 6

Aim Determination of Young's Modulus and bending moment of a beam by using travelling microscope by the method of flexure.

Apparatus Two uprights with knife edge, 1 m long metallic rectangular bar, hanger, measuring weights, spirit level, travelling microscope, magnifying glass, table lamp etc.

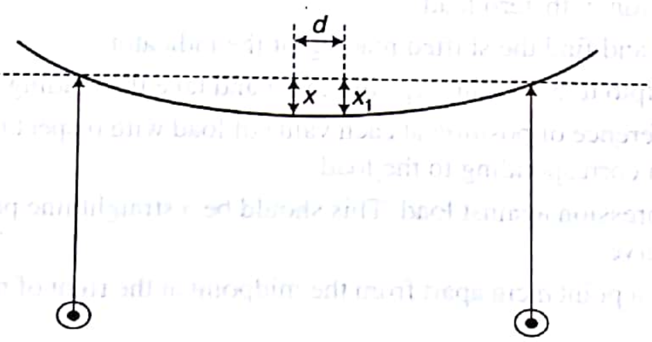


Fig. D.6 Experimental bar on knife edges of two uprights

Theory The experimental bar is shown on the knife edges of two uprights in the Fig. D.6. If one attaches a load of mass m to the hanger, the depression of the bar is given by

$$x = \frac{mg\,l^3}{4bd^3Y}$$

$$\Rightarrow \qquad Y = \frac{mgl^3}{4bd^3x}$$

where Y = Young's modulus of the material of the bar (dyne/cm^2),
l = length of the bar between the two knife edges,
g = acceleration due to gravity,
b = breadth of the bar (cm) and
d = depth of the beam in cm.

The bending moment is given by

$$M_b = \frac{YI_g}{R} = \frac{1}{12R}(Y \times b \times d^3) \quad \text{(where } I_g \text{ is the moment of inertia of the bar)}$$

where

$$R = \frac{d^2}{2\Delta x} - \Delta x \approx \frac{d^2}{2\Delta x}$$

and d is the distance between the midpoint of the bar and the point concerned where the bending moment is to be calculated.

$$\Delta x = x \sim x_1$$

where x_1 is the depression at d.

Procedure

1. Measure the breadth and depth of the rectangular bar with the help of slide callipers.
2. The midpoint of the bar is marked as O and each side of it is marked at an interval of 10 cm.
3. Place the knife edges under the 50 cm mark at both sides of the midpoint. The length of the bar between the knife edges (l) is 1 metre.
4. Focus the travelling microscope on the indicator of the hanger. Take the reading on the vertical scale. This is the depression with zero load.
5. Apply 0.5 kg load and find the shifted reading of the indicator.
6. Increase the load upto to 3.5 kg in steps of 0.5 kg and take the reading of the indicator in each case.
7. Determine the difference of position at each value of load with respect to that of zero load. This gives you the depression corresponding to the load.
8. Plot a graph of depression against load. This should be a straight line passing through the origin. Get the slope of this curve.
9. Place the hanger at a point d cm apart from the midpoint at the right of it. Take two readings with zero and 1 kg load.

Measurements

Length of the bar l = ... cm
Breadth of the bar b = ... cm
Depth of the bar d = ... cm
Least count of Tr. microscope scale = ... cm

Observations

Table D.6a Measurement of depression for Young's modulus calculation

S. No.	Load M (kg)	Travelling Microscope readings						Mean	Depression
		During increasing load			During decreasing load				
		MSR (cm)	VSR	TR (cm)	MSR (cm)	VSR	TR (cm)	(cm)	x (cm)
1									
2									
3									
4									
5									
6									
7									

Table D.6b Measurement of depression for calculation of bending moment

Sl. No.	Load (kg)	Reading of centre MSR VSR Total	Depression x	Reading of right MSR VSR Total	Depression x_1
1	0				
2	1				

$$\Delta x = x_1 \sim x$$

$$R = \frac{d^2}{2\Delta x}, \quad \text{Bending moment } M_b = \frac{1}{12R}(Y \times b \times d^3)$$

Results

Percentage of Error

Discussions

EXPERIMENT NO. 7

Aim Determination of modulus of rigidity of the material of a rod by statical method.

Apparatus Rigidity measuring apparatus, screw-gauge, slide calipers, weights, metre scale.

Theory The modulus of rigidity of the material of the rod is given by

$$\eta = \frac{180\, ldg}{\pi^2 r^4}\left[\frac{m}{\phi}\right]$$

where, l = effective length rod = $l_2 - l_1$

g = acceleration due to gravity

d = diameter of the pulley
r = radius of the rod
m = mass of the weight placed at pan
ϕ = angular twist = $\theta_1 \sim \theta_2$

The effective length l of the rod is the distance between two marked positions l_1 and l_2. And ϕ is the twist of the length l which is the difference of the twists of the positions l_1 and l_2 given by θ_1 and θ_2 respectively.

Procedure

1. Measure the radius of the rod with the help of a screw gauge.
2. Measure the diameter of the pulley with the help of a slide calipers.
3. Note down the value of the effective length from the scale attached.
4. Put weights in the pan in steps of 0.5 kg. Determine values of θ_1 and θ_2 from the protractors attached to the rod. Similarly take reading while unloading. Take the average of loading and unloading values.
5. Repeat steps 3 and 4 for another value of effective length.
6. Draw a graph of the angular twist versus load for each value of effective length.

Measurement

Effective length of the rod l= ... cm

Diameter of the rod $2r$ = ... cm

Diameter of the pulley $\frac{d}{\lambda}$ = ... cm

Observations

Table D.7a Measurement of twist for $l = x$ cm

No. of Obs.	Load (kg)	Load increasing (degree)			Load decreasing (degree)			$\theta_1 \sim \theta_2$ (degree)
		θ'_1	θ''_1	$\theta_1 = \frac{\theta'_1 + \theta''_1}{2}$	θ'_2	θ''_2	$\theta_2 = \frac{\theta'_2 + \theta''_2}{2}$	
1								
2								
⋮								
⋮								
8								

Table D.7b Measurement of twist for $l = y$ cm

No. of Obs.	Load (kg)	Load increasing (degree)			Load decreasing (degree)			$\theta_1 \sim \theta_2$ (degree)
		θ'_1	θ''_1	$\theta_1 = \frac{\theta'_1 + \theta''_1}{2}$	θ'_2	θ''_2	$\theta_2 = \frac{\theta'_2 + \theta''_2}{2}$	
1								
2								
⋮								
⋮								
8								

Graph Plot two graphs for two values of l. Plot $\phi(=\theta_1 \sim \theta_2)$ versus load.

Calculations

Percentage Error

Discussions

EXPERIMENT NO. 8

Aim To determine the modulus of rigidity of the material of a wire by dynamical method.

Apparatus A torsional pendulum, a cylindrical mass, slide calipers, screw-gauge, stop watch.

Theory The time period of oscillation of a trosional pendulum is given by

$$T = 2\pi\sqrt{\frac{I}{C}} \qquad ...(1)$$

where I is the moment of inertia of the pendulum-bob, C is the moment of the torsional couple per unit angle which again can be expressed as

$$C = \frac{\eta \pi r^4}{2L} \qquad ...(2)$$

where η is the modulus of rigidity of the material of the wire of the torsional pendulum, r is its radius and L is the effective length. For a cylindrical bob the moment of inertia is given by

$$I = \frac{1}{2}MR^2 \qquad ...(3)$$

where M and R the mass and radius of the cylinder.

From equation (1), (2) and (3) the modulus of rigidity can be determined as

$$\eta = \frac{8\pi L}{T^2 r^4}\left(\frac{1}{2}MR^2\right)$$

where $T^2 = T_1^2 \sim T_2^2 =$ The difference of the squares of the time periods with and without putting the cylindrical mass on the platform of the pendulum.

Procedure

1. Measure the radius of the wire at several places with the help of a screw gauge.
2. Measure the effective length of the pendulum.
3. Measure the diameter of the cylindrical bob with the help of a slide calipers.
4. Mass of the cylinder is usually supplied.
5. Rotate the cylindrical bob at a small angle about the suspension wire having put it on the pan and then leave it to oscillate freely. This will initiate a torsional oscillation in the bob. Measure the time period of this oscillation (T_1) with the help of a stop watch.

6. Repeat the Step 5 without putting the bob on the platform. Measure the time period T_2.
7. Repeat Steps 5 and 6 two other times.

Measurements

Diameter of the suspension wire $d = 2r = ...$

Length of the suspension wire $l = ..$

Mass of the cylinder $M = ...$

Observations

Table D.8a Time period of oscillation (with the cylinder on the pan).

No. of Obs.	*No. of oscillations observed (n)*	*Time taken for n oscillation t (second)*	*Time period T = t/n (second)*	*Mean time period (second)*
1				
2				
3				

Table D.8b Time period of oscillation without the cylinder on the pan

No. of Obs.	*No. of oscillations observed (n)*	*Time for n oscillation t (second)*	*Time period T = t/n (second)*	*Mean time period (second)*
1				
2				
3				

Results

Percentage of Error

Discussions

EXPERIMENT NO. 9

Aim To determine the dispersive power of the material of the given prism.

Apparatus Spectrometer, prisms, mercury vapour lamp, spirit label.

Theory The dispersive power of the material of a prism is given by

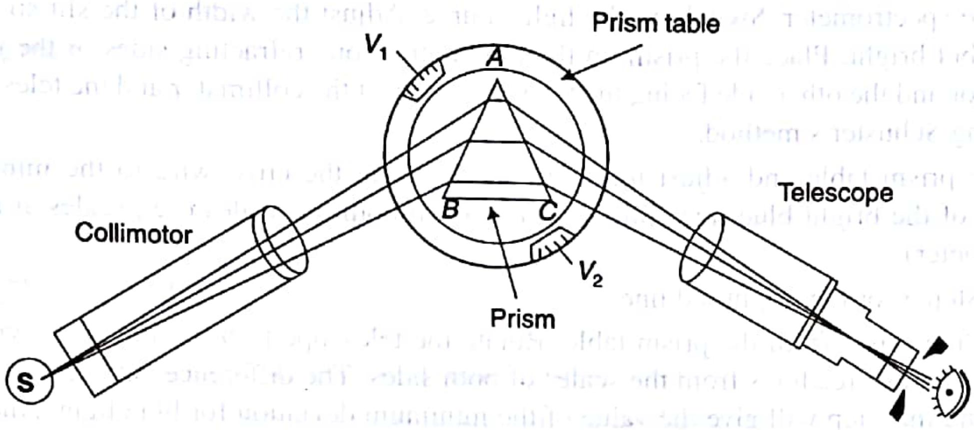
$$P_d = \frac{\mu_b - \mu_r}{\mu - 1}$$

where,

μ_b = refractive index of the material of the prism for blue colour.

μ_r = refractive index of the material of the prism for red colour.

μ = average of μ_b and μ_r.



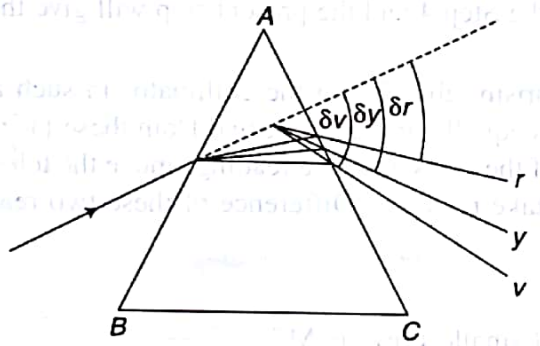


Fig. D.9(a) Experimental set up of dispersive power of the material of a prism. (b) Angle of minimum deviation of red, yellow and violet light.

Refractive index for blue colour is given by

$$\mu_b = \frac{\sin\left(\dfrac{A + \delta_{mb}}{2}\right)}{\sin (A/2)}$$

and that for red colour is given by

$$\mu_r = \frac{\sin\left(\dfrac{A + \delta_{mr}}{2}\right)}{\sin (A/2)}$$

where, δ_{mb} and δ_{mr} are the angles of minimum deviation for blue and red colour respectively and A is the angle of the prism.

So, the dispersive power of the material of the prism is given by

$$P_d = \frac{\mu_b - \mu_r}{\dfrac{\mu_b + \mu_r}{2} - 1}$$

Procedure

1. Measure the vernier constant of the spectrometer.
2. Level the spectrometer. Switch on the light source. Adjust the width of the slit so that it is narrow enough but bright. Place the prism on the prism table, one refracting sides of the prism facing the collimator and the other side facing the telescope. Adjust the collimator and the telescope for parallel rays using Schuster's method.
3. Turn the prism table and adjust the telescope to bring the cross-wire to the minimum deviation position of the bright blue line. Take readings from both the scales (i.e., scales at two sides of the spectrometer).
4. Repeat Step 3 for the bright red line.
5. Remove the prism from the prism table. Rotate the telescope to bring the cross-wire on the direct light beam. Take readings from the scales of both sides. The difference between the readings in the Step 3 and this step will give the value of the minimum deviation for blue light. And the difference between the readings in the Step 4 and the present step will give the minimum deviation for the red light.
6. Place the prism on the prism table facing the collimator in such a way that the light rays fall on both the transparent faces equally and get reflected from these faces. Rotate the telescope to get the reflected rays from one of the faces and take readings move the telescope to receive the reflected ray from the other face and take readings. Difference of these two readings will be equal to twice the angle of the prism.

Observations

$$\text{Vernier constant} = \frac{\text{1 smallest div. in. MS}}{\text{No. of div. in VS}}$$

Table D.9a Determination of minimum deviation angle

Colour of light	*Vernier – I*				*Vernier – II*				*Mean*
	MSR (unit)	*VSR*	*Total V_1 (unit)*	*$\delta_m = V_1 - V_{01}$ (unit)*	*MSR (unit)*	*VSR*	*Total V_2 (unit)*	*$\delta_m = V_2 - V_{02}$ (unit)*	*(unit)*
Blue									
Red									
Direct				V_{01}				V_{02}	

Table D.9b Determination prism angle

No. of vernier	*Reading with telescope on left side (R_1)*			*Reading with telescope on right side (R_2)*			*$2A = R_1 \sim R_2$*	*Mean 2A*
	MSR (unit)	*VSR*	*Total (unit)*	*MSR (unit)*	*VSR*	*Total (unit)*	*unit*	*unit*
Vernier I								
Vernier II								

Calculation

Percentage Error

Discussions

EXPERIMENT NO. 10

Aim To determine the wavelength of sodium light by using Fresnel's bi-prism.

Apparatus Optical bench with four uprights, sodium vapour lamp, bi-prism, convex lens, slit, eye-piece with a micrometer.

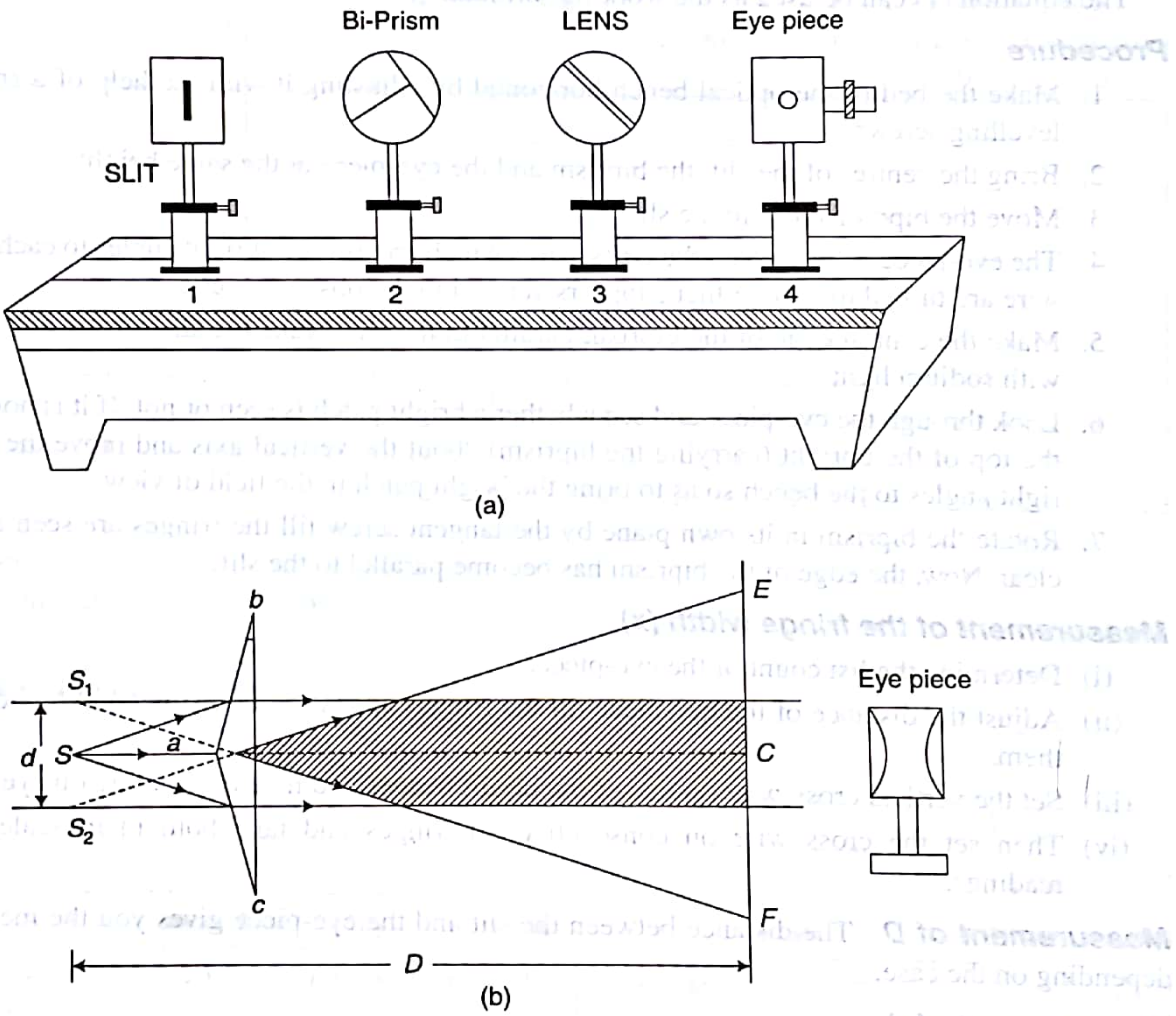


Fig. D.10 (a) The optical bench; (b) Formation of fringes with a bi prism

Theory Light rays from sources (Fig. D.10(b)) appear to be diverging from the two halves of the Fresnel's biprism leading to the formation of two virtual images S_1 and S_2. Light rays from the two virtual images interfare on the focal plane and form alternating dark and bright fringes. And the fringe-width is given by

$$x = \frac{D\lambda}{d}$$

or,
$$\lambda = \frac{dx}{D} \quad ...(1)$$

where, λ is the wavelength of sodium light, D is the distance between the virtual source S_1 and S_2 and the screen (i.e., the eye-piece of the micrometer) and d is the distance between S_1 and S_2.

If a convex lens is placed between the eye-piece and the biprism and its position is changed then for two positions real images of the virtual sources can be seen through the eye-piece. In these two positions, if the separations between the real images are d_1 and d_2, then one can have,

$$d = \sqrt{d_1 d_2} \quad ...(2)$$

The equation (1) can be used as the working formula.

Procedure

1. Make the bed of the optical bench horizontal by adjusting it with the help of a spirit level and the levelling screws.
2. Bring the centres of the slit, the biprism and the eye-piece at the same height.
3. Move the biprism close to the slit.
4. The eye-piece is focused on the cross-wires which are usually at right angles to each other. The cross-wire are turned till one of them appears vertical to the observer's eye.
5. Make the central edge of the biprism parallel to the slit. Make the slit narrow and get it illuminated with sodium light.
6. Look through the eye-piece and see whether a bright patch is seen or not. If it is not seen, then rotate the top of the upright (carrying the biprism) about the vertical axis and move the upright a little at right angles to the bench so as to bring the bright patch in the field of view.
7. Rotate the biprism in its own plane by the tangent screw till the fringes are seen and become quite clear. Now, the edge of the biprism has become parallel to the slit.

Measurement of the fringe width (x)

(i) Determine the list count of the eye-piece.

(ii) Adjust the distance of the eye-piece till the fringes appear quite distinct and broad enough to count them.

(iii) Set the vertical cross-wire on one of the bright fringes and note the reading on eye-piece scale.

(iv) Then set the cross wire on consecutive 20 fringes and take both main scale and micrometer readings.

Measurement of D The distance between the slit and the eye-piece gives you the measure of D_1 or D_2 depending on the case.

Measurement of d

(i) For this part, keep the distance between the eye-piece and the slit slightly more than four times the focal length of the convex lens. Do not alter the positions of the slit and the biprism.

(ii) Introduce the convex lens between the biprsim and the eye-piece and place it near the eye-piece. Move the lens towards the biprism till two sharp images of the slit are seen.

(iii) Measure the distance d_1 between two images of the slit by the eye-piece micrometer.

(iv) Move the lens towards the biprism till two images are seen again.

(v) Measure the distance d_2 between these two images.

(vi) For measurement of d_1 and d_2 use a lens of good quality so that the images be free from chromatic defects.

(vii) Carry out the measurement of d_1 and d_2 with great accuracy because slight error in this measurement considerably affects the result.

Observations Least count of the micrometer is given by

$$\text{LC} = \frac{\text{1 div. of micrometer}}{\text{No. of div. of the CS}} = \frac{l}{n}$$

Table D.10a Measurement of fringe width x.

No. of fringe s	*Micrometer reading (a)*			*No. of fringe s*	*Micrometer reading (b)*			*Width for 10 fringe x (a ~ b) (cm)*	*Mean width for 10 fringe W (cm)*	*One Fringe width x = (W/10) (cm)*
	M.S.	*V.S.*	*Total (a) (cm.)*		*M.S.*	*V.S.*	*Total (b) (cm)*			
1				11						
2				12						
3				13						
4				14						
5				15						
6				16						
7				17						
8				18						
9				19						
10				20						

Measurement of D

Position of upright carrying slit = ... cm

Position of upright carrying eye-piece = ... cm

Observed value of D: cm

Table D.10b Measurement of $d = \sqrt{d_1 d_2}$

SL. No.	*Micrometer reading*															*d = $\sqrt{d_1 d_2}$ cm.*	*mean d cm*
	1st position of lens							*2nd position of lens*							*d_2 cm*		
	Image 1			*Image 2*			*d_1*	*Image 1*			*Image 2*						
	M.S	*V.S*	*Total (cm)*	*M.S*	*V.S.*	*Total (cm)*	*cm*	*M.S*	*V.S*	*Total (cm)*	*M.S*	*V.S.*	*Total (cm)*				
1.																	
2.																	

Calculations

Percentage of Error

Discussions

APPENDIX E

Useful Physical Constants

Rest mass of electron	m_e	9.1×10^{-31} kg
Charge of electron	e	-1.6×10^{-19} c
Specific charge of electron	e/m_e	1.758×10^{-11} c/kg
Rest mass of proton	m_p	1.673×10^{-27} kg
Charge of proton	e_p	$+ 1.6 \times 10^{-19}$ c
Speed of light in vaccum	c	3×10^{8} m/s
Planck's constant	h	6.63×10^{-34} Js
Boltzmann constant	k	1.38×10^{-23} J/K
Universal gas constant	R	8.314 J/(k mol)
Gravitational constant	G	6.6720×10^{-11} Nm^2/kg^2
Avogadro's number	N_A	6.023×10^{23} mol^{-1}
Refractive index of water	μ_w	1.33
Refractive index of glass	μ_g	1.50
Viscosity of water (20°C)	η_w	1.002×10^{-3} Ns/m^2
Standard atmospheric pressure	P_S	1.013×10^{5} Nm^{-2}
Bohr radius	r	5.29×10^{-11} m = 0.53 Å
Compton wavelength of electron	λ_c	2.42×10^{-12} m
Permittivity of free space	ε_0	8.85×10^{-12} F/m
Permeability of free space	μ_0	$4\pi \times 10^{-7}$ H/m
Slefan's radiation constant	σ	5.67×10^{-8} w/m^2 K^4
Rydberg constant	R_d	1.09737×10^{7} m
Young's modulus of steel	Y_S	2.1×10^{11} Pa

Model Question Papers

MODEL QUESTION PAPER – 1

Group A

Multiple-Choice Type Questions

1. Choose the right alternatives for any ten of the following:

(i) For a particle executing S.H.M. the phase difference between displacement and velocity

(a) π (b) 0 (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

(ii) The quality factor of a series L-C-R circuit is

(a) $\frac{1}{\sqrt{RLC}}$ (b) $\frac{1}{R}\sqrt{\frac{L}{C}}$ (c) $\sqrt{\frac{RL}{C}}$ (d) $\sqrt{RLC}$

(iii) Two sources will be coherent if they

(a) have constant wavelength (b) have a constant phase difference

(c) have a constant amplitude (d) none of these

(iv) The resolving power of a grating, having N number of total rulings in n^{th} order is

(a) $\frac{n}{N}$ (b) nN (c) $\frac{N}{n}$ (d) none of these

(v) Optic axis is a direction along which

(a) the ordinary ray travels faster than the extraordinary ray

(b) the extraordinary travels faster than the ordinary ray

(c) both the extraordinary and the ordinary ray travel with same velocity

(d) none of these

(vi) In lasing action, the spontaneous emission does not depend on

(a) the number of atoms present in the excited state

(b) the intensity of the incident light

(c) both of them

(d) none of these

(vii) If a hologram breaks into pieces, each piece can reproduce

(a) the entire image (b) the half image (c) one third image (d) one fourth image

(viii) The de Broglie wavelength of a moving electron subjected to a potential V is

(a) $\frac{1.26}{\sqrt{V}}$ Å (b) $\frac{12.26}{\sqrt{V}}$ Å (c) $\frac{12.26}{V}$ Å (d) $\frac{12.26}{\sqrt{V}}$ Å

(ix) The wavelength of which the spectral energy density of emitted radiation at temperature T from a black body attains maximum value proportional to

(a) $\frac{1}{T}$ (b) T (c) T^4 (d) $T^{3/2}$

(x) Origin of continuous x-rays is due to the process of

(a) ionisation (b) inner orbital transition

(c) bremsstrahllung (d) none of these

(xi) The number of Bravais lattices are

(a) 12 (b) 14 (c) 13 (d) 10

(xii) In the recording of the hologram one superimposes

(a) object wave and reference wave (b) two object waves

(c) two reference waves (d) none of these

(xiii) The product of group velocity (v_g) and phase velocity (v_p) of de Broglie waves is

(a) c (b) c^2 (c) less than c (d) none of these

(xiv) Uncertainty principle is the consequence of

(a) wave nature of particle (b) wave particle duality

(c) particle nature of wave (d) particle-particle interaction

Group B

Short-Answer Type Questions

Answer any three of the following

2. Write down the differential equation of a series *LCR* circuit driven by a sinusoidal voltage. Identify the natural frequency of this circuit. Find out the condition that this circuit will show an oscillatory decay and find out the relaxation time.

3. (a) Define temporal and spatial coherence.

   (b) In a Newton's ring experiment the diameter of the 4^{th} and 12^{th} rings are 0.4 cm and 0.7 cm respectively. Find the diameter of the 20^{th} dark ring. [0.908 cm]

4. (a) What is population inversion? Explain

   (b) What are the basic principle of holography?

5. (a) Show that in cubic crystal of side a the interplaner spacing between consecutive parallel planes of Miller indices (hkl) is

$$d_{hkl} = \frac{a}{\sqrt{h^2 + k^2 + l^2}}$$

   (b) Can x-ray be diffracted from a single slit of width 0.1 mm? Explain.

6. Draw a neat diagram of ruby laser. Discuss the operation of a ruby laser with the help of energy diagram.

Group C

Long-Answer Type Questions

Answer any three questions

7. (a) In a Young's double-slit experiment, the angular width of a fringe found on a distant screen is 0.1°. The wavelength of light used is 6000 Å. What is the spacing between the slits? [3.44×10^{-2} cm]
   (b) Show that interference phenomenon does not violate the principle of conservation of energy.
   (c) What is meant by the resolving power of an optical instrument?
   (d) Calculate the minimum number of lines in a grating which will just resolve the lines of wavelengths 5890 Å and 5896 Å in the second order. [491]
   (e) Discuss the phenomenon of double refractions.
8. (a) Write down Planck's radiation law and hence obtain Wien's displacement law.
   (b) Why do unmodified lines appear in compton scattering?
   (c) Show that a free electron at rest cannot absorb a photon.
   (d) Show that the de Broglie wavelength for a material particle of rest mass m_0 and charge q associated from rest through a potential difference of V volts relativistically is given by

$$\lambda = \frac{h}{\left[2m_0 qV\left(1 + \frac{qV}{2m_0 c^2}\right)\right]^{1/2}}$$

9. (a) Describe the Davisson and Germer experiment giving emphasis on the interpretation of the result of the experiment.
   (b) The maximum uncertainty in the position of an electron in a nucleus is 2×10^{-14}m. Find the minimum uncertainty in its momentum, given $h = 6.63 \times 10^{-34}$ JS. [212.6×10 kg ms^{-1}]
   (c) What do you mean by Bravais lattice? Find the Miller indices of a plane which intercepts at $a, \frac{b}{2}, 3c$ in a simple cubic unit cell.
   (d) Starting from de Broglie's hypothesis show that the group velocity associated with particle is the same as the particle velocity.
10. (a) Define Einstein's A and B co-efficients and deduce their mutual relation.
   (b) If the wavelength of radiation of a ruby laser is 694.3 nm, what is the energy of the photon emitted? [1.787 eV]
   (c) Define metastable state.
   (d) Why is laser needed for holography?
   (e) State some of the applications of holography.
11. (a) Define free vibration. Can a real vibrator vibrate completely freely. If not why?
   (b) What is the difference between amplitude resonance and velocity resonance?
   (c) What is the vital-factor (quality factor) of a forced oscillator?

(d) A voltage having r.m.s value $V_{rms} = 100$ volts is applied to a series resonant circuit with resistance $R = 10\ \Omega$, inductance $L = 10$ mH and capacitance $C = 1\ \mu F$. Calculate the natural frequency and current at resonance in the circuit.

(e) If $y = f_1(x - vt) + f_2(x + vt)$ where f_1 and f_2 are the two functions then show that y satisfies wave equation.

MODEL QUESTION PAPER – 2

Group A

Multiple-Choice Type Questions

1. Choose the right alternatives for any ten of the following:

(i) The time period (T) of a damped vibration is

(a) $\dfrac{2\pi}{\sqrt{2k^2 - w_0^2}}$ (b) $\dfrac{2\pi}{\sqrt{w_0^2 - k^2}}$ (c) $2\pi\sqrt{w_0^2 - k^2}$ (d) none of these.

(ii) If visible light is used to study the Compton effect then Compton shift will be

(a) negative (wavelength of the scattered light will be lesser)

(b) more positive than what is observed with X-ray

(c) zero

(d) positive but not detectable in the visible window.

(iii) Origin of continuous X-ray is due to the process of

(a) ionisation (b) inner orbital transition

(c) bremsstrahlung (d) none of these.

(iv) In a ruby laser, population inversion is achieved by

(a) optical pumping (b) inelastic atom-atom collisions

(c) chemical reaction (d) applying strong electric field.

(v) A stone is dropped from the top of a building. What happens to the de Broglie wavelength of the stone as it falls?

(a) it increases (b) it decreases (c) remains constant (d) de Broglie

(vi) Assuming that the atoms in crystal are spheres of equal size and touching each other, it can be shown that atomic radius of bcc is equal to

(a) $\dfrac{a}{2}$ (b) $\sqrt{3}\,\dfrac{a}{4}$ (c) $\sqrt{3}\,\dfrac{a}{2}$ (d) $\dfrac{a}{2\sqrt{2}}$

(vii) Holography is a method in which one

(a) record the amplitude only

(b) records the phase only

(c) not only records the amplitude but also the phase of the light wave

(d) none of these.

(viii) Davision and Germer studied electron diffraction with nickel crystal and found a first-order peak at 65° with electron beam of 54 eV. If, instead a 216 eV beam were used them the peak would have been at

(a) 27° (b) 54° (c) 130° (d) 260°

(ix) A light ray is passing through a calcite crystal. If the plane of vibration of the light ray is perpendicular to the optical axis then the ray
(a) is an *O*-ray (b) is an *E*-ray
(c) has both non-zero *O* and *E* components (d) cannot propagate through the crystal.

(x) The colour of the laser output in case of ruby laser.
(a) violet (b) blue (c) red (d) green

(xi) The fringes in Young's double slit experiment are
(a) equally spaced (b) not equally spaced
(c) partially equally spaced (d) none of these.

(xii) Missing order in the interference maxima in Fraunhofer double-slit pattern occurs if the
(a) slit width is decreased
(b) slit width is constant, but the slit separation is increased
(c) slit width increased
(d) none of these.

(xiii) Wien's displacement law could explain the distribution of thermal radiation in the spectrum of black body radiation in the region of
(a) higher wavelength (b) lower wavelength
(c) middle wavelength (d) none of these.

(xiv) Uncertainty principle has significance only in
(a) macroscopic world (b) microscopic world
(c) both macroscopic and microscopic world (d) none of these.

Group B

Short-Answer Type Questions

Answer any three of the following

2. Establish the differential equation of simple harmonic motion. A point mass m is suspended at the end of a massless wire of length l and cross-sectional area A. If the Young's modulus of elasticity of the wire be Y then obtain the frequency of oscillation for the simple harmonic motion along the vertical line.
3. Do interference phenomena violate the principle of conservation of energy? What happens to fringe width if the double slit set up is immersed in water?
4. In a lasing process, the ratio of population of two energy states E_1 and E_2 (E_2 being a metastable state) is $1{:}1.009 \times 10^{25}$. Calculate the wavelength of the laser beam at a temperature of 320 K.
5. A beam of x-rays of wavelengths 0.842 Å is incident on a crystal at a glancing angle of 8° 35′, when first-order Bragg's reflection occurs. Calculate the glancing angle for the third-order reflection.
6. Derive an expression for Compton shift in wavelength for a photon scattered from a free electron at an angle θ.

Group B

Short-Answer Type Questions

Answer any three of the following

2. Starting from the equation of plane progressive wave in one dimension, derive the differential equation of the wave.
3. In a He-Ne laser transition from 3S to 2P level gives a laser emission of wavelength 632.8 nm. If the 2P level has energy equal to 15.2×10^{-19} J, calculate the pumping energy required. Assume there is no loss.
4. If the energy levels of K, L and M state of Pt are 78 KeV, 12 KeV and 3 KeV respectively. Calculate the wavelength K_α and K_β lines emitted from platinum.
5. What is the minimum number of lines of a grating which resolves the 3rd order spectrum of two lines having wavelengths of 5890 Å and 5896 Å.
6. What is the minimum kinetic energy of the recoil electron?
   Calculate the maximum energy transferred by the photon to the electron. Is it possible for a photon to transfer its whole energy to the electron?

Group C

Long-Answer Type Questions

7. (a) In a one-dimensional motion a mass 10 gm is acted on by a restoring force 10 dynes/cm and a restoring force 2 dynes/cm. Find
   (i) whether the motion is a periodic or oscillatory.
   (ii) the resistance force per unit velocity which will make the motion critically damped.
   (iii) the mass for which the given forces will make the motion critically damped.
   (b) 9 kg of hg is poured into a U-tube of uniform internal diameter of 1.2 cm. It oscillates freely about its equilibrium position. Calculate the period of oscillation.
8. (a) Write down short note on Bravais lattice and Miller indices.
   (b) What do you mean by x-ray spectrum? Draw continuous x-ray spectra for different applied voltages.
   (c) An x-ray tube is operated at 50 kV. Find the maximum speed of the electron striking the surface of the anode and the shortest wavelength of x-ray produced.
9. (a) What is grating element? Find out the relation of grating element with the number of rulings.
   (b) What is the Rayleigh criterion of resolution?
   (c) Discuss the phenomena of double refractions.
   (d) A ray of light is incident on a glass of refractive index 1.732 at the polarization angle. Find (i) angle of polarization and (ii) the angle of refraction of the ray.
   (e) State Malus law.

10. (a) Write down Planck's radiation law and hence prove Wien's law.
(b) Show graphically, how the energy density versus frequency plot of black body radiations is changed if the temperature is increased.
(c) Describe briefly the Davisson – Germer experiment. What inferences may we draw from this experiment?

MODEL QUESTION PAPER – 4

Group A

Multiple-Choice Type Questions

1. Choose the right alternatives for any ten of the following

(i) Two mutually perpendicular SHMs with equal time periods but different amplitudes are superposed. If the phase difference between these oscillations is 45°, then they form a

(a) circle (b) straight line (c) ellipse (d) parabola

(ii) The centre of the Newton's rings for the transmitted system of a monochromatic source of light is

(a) dark (b) partially dark (c) bright (d) none of these

(iii) A plane polarized light of intensity I_o is incident on a polarizer. whose axis of polarization is 30° with respect to the direction of propagation of the incident light. The intensity of light coming out of the polarizer is

(a) I_o (b) $\frac{\sqrt{3}\, I_o}{2}$ (c) $\frac{3\, I_o}{4}$ (d) $\frac{I_o}{4}$

(iv) If λ_L and λ_K are wavelength of L and K x-rays respectively, then

(a) $\lambda_L < \lambda_K$ (b) $\lambda_L > \lambda_K$ (c) $\lambda_L = \lambda_K$ (d) $\lambda_L = 4\lambda_K$

(v) The nearest neighbour distance in the case of BCC structure is

(a) $\frac{a\sqrt{3}}{3}$ (b) $\frac{a\sqrt{2}}{3}$ (c) $\frac{2a}{\sqrt{3}}$ (d) $\frac{2a}{\sqrt{2}}$

(vi) The Compton wave length is

(a) $\frac{h}{m_0 c}$ (b) $\frac{h}{m_0 c^2}$ (c) $\frac{h\, m_0}{c}$ (d) $\frac{h}{m_0 c^3}$

(vii) The absorptive power of a black body is

(a) 1 (b) 0 (c) 2 (d) 3

(viii) In He-Ne laser, the laser light is emited due to the transition from

(a) $3s \rightarrow 2p$ (b) $3s \rightarrow 3p$ (c) $2s \rightarrow 2p$ (d) none of these

(ix) The reconstruction process in holography produces

(a) a virtual image (b) a real image
(c) both virtual and real image of the object (d) none of these

(x) Phase velocity and group velocity are equal when the medium is
   (a) dispersive   (b) isotropic   (c) non-dispersive   (d) none

(xi) For heavy particle
   (a) $\Delta x\, \Delta v_x = \frac{\lambda}{2}$ (b) $\Delta x\, \Delta v_x = 0$ (c) $\Delta x\, \Delta v_x = \alpha$ (d) $\Delta x\, \Delta v_x = \frac{\hbar}{2}$

(xii) The intensity of principle maxima in the spectrum of a grating with N number of lines is proportional to
   (a) $\frac{1}{N}$ (b) N (c) N^2 (d) $\frac{1}{N^2}$

Group B

Short-Answer Type Questions

Answer any three of the following

2. What are sharpness of resonance and Q-factor? Show that the Q-factor is given by

$$Q = \frac{\omega_0 L}{R}$$

in case of series LCR resonant circuit.

3. Sodium Light of wavelength 589 nm and 589.6 nm are made incident normally on a grating having 500 lines per mm. Calculate the angular dispersion of these lines in the spectrum of first order.
4. State and explain Brewster's law of polarization clearly indicating the nature of polarization of the reflected and refracted ray.
5. An electron has de Broglie wavelength of 0.15 Å. Compute the phase and group velocities of the de Broglie waves. Find the kinetic energy of the electron.
6. Find the interplanar distance of (110) plane and (111) plane of Nickel crystal. The radius of Nickel atom is 1.245 Å.

Group C

Long-Answer Type Questions

Answer any three questions

7. (a) A particle is subject to a harmonic restoring force and a damping force. Its equation of motion is given by

$$m\frac{d^2x}{dt^2} = -sx - k\frac{dx}{dt}$$

Under the condition of critical damping, find the expression for displacement as a function of time.

(b) What do you understand by 'logarithmic decrement', 'relaxation time' and 'quality factor' of a weakly damping oscillator?

(c) Do interference phenomena violate the principle of conservation of energy?

8. (a) What are the conditions for sustained interference pattern?

(b) Why two independent source of light of same wavelength cannot produce interference pattern?

(c) Show that intensity distribution for diffraction in a single slit is given by

$$I = I_o\left(\frac{\sin^2\beta}{\beta^2}\right) \quad \left[\text{where } \beta = \frac{\Pi d\sin\theta}{\lambda}\right]$$

where the symbols have their usual significance. Deduce also conditions for maxima and minima.

9. (a) Why do you need population inversion in a laser?

(b) Obtain a relation between Einstein's A and B co-efficients. What are their physical significances?

(c) Define holography and hologram.

(d) What are object wave and reference wave?

10. (a) The energy gap in Ge is 0.72 eV. Find the maximum wavelength of a radiation which can cause transition from the valence band to conduction band.

(b) Derive an expression for Compton shift in wavelength for a photon scattered from a free electron at an angle θ. At which angle will the shift be maximum?

(c) Show that the de Broglie wavelength of a particle of mass m and kinetic energy E_K is given by

$$\lambda = \frac{hc}{\sqrt{E_K(E_K + 2m_0c^2)}}$$

MODEL QUESTION PAPER – 5

Group A

Multiple-Choice Type Questions

1. Choose the right alternatives for any ten of the following.

(i) Example of a weakly damped harmonic oscillator is

(a) dead-beat galvanometer

(b) tangent galvanometer

(c) ballistic galvanometer

(d) discharge of a charged capacitor through a resistance.

(ii) For constructive interference, the phase difference is an even multiple of

(a) $\frac{\pi}{2}$ (b) 2π (c) π (d) none of these

(iii) In Fraunhofer diffraction, the incident wavefront is

(a) plane (b) spherical (c) cylindrical (d) arbitrary shape

(iv) When two parellel rays of x-rays of wavelength λ, are incident at an angle θ on a crystal with lattice separation d, constructive interference would be observed when (n is an integer)

(a) $n\lambda = 2d \sin \theta$ (b) $n\lambda = d \sin \theta$ (c) $n\lambda = d \sin 2\theta$ (d) $n\lambda = 2d \sin 2\theta$

(v) Miller indices of a plane which has intercepts of 2, 3 and 4 units along the three axes are

(a) (6, 4, 3) (b) (2, 3, 4) (c) (3, 2, 1) (d) (2, 3, 2)

(vi) All the radiation laws can be shown to be special cases of

(a) Wien's law (b) Stefan–Boltzmann's law

(c) Rayleigh–Jeans law (d) Planck's law.

(vii) Uncertainty principle is the consequence of

(a) wave nature of particle (b) wave-particle duality

(c) particle nature of wave (d) particle-particle interaction.

(viii) For holography system the exposure time is of the order of

(a) 5 seconds and depends on the colour of the object

(b) nearly 2 seconds

(c) 1 second

(d) 50 seconds

(ix) The wavelength of He-Ne laser is

(a) 632.8 nm (b) 600 nm (c) 532.8 nm (d) 500 nm

(x) Number of optic axes in a uniaxial crystal is

(a) one (b) two (c) five (d) ten

(xi) If λ_r and λ_{nr} be the relativistic and non–relativistic wavelength of the electron, then

(a) $\lambda_r > \lambda_{nr}$ (b) $\lambda_{nr} = \lambda_r$ (c) $\lambda_{nr} > \lambda_r$ (d) $\lambda_r = \dfrac{1}{\lambda_{nr}}$

(xii) The atomic packing factor for BCC structure is

(a) 74% (b) 68% (c) 52% (d) none of these

Group B

Short-Answer Type Questions

Answer any three of the following

2. A cubical block of side L cm and density d is floating in a water of density ρ ($\rho > d$). The block is slightly depressed and released. Show that it will execute SHM and hence determine the frequency of oscillation.

3. In an interference experiment d is the distance between the two coherent sources of light with wavelength λ and D is the source screen distance. Show that the separation between the two consecutive dark bands is given by

$$\beta = \frac{D\lambda}{d}$$

4. Calculate the ratio of the stimulated emission to the spontaneous emission at a temperature 300 K for the sodium D line.

5. Determine the kinetic energy of a particle for which its de Broglie wavelength becomes equal to its Compton wavelength.
6. X-rays with 1.54 Å are used for the calculation of the d_{100} plane of a cubic crystal. The Bragg's angle of first order reflection is 10°. What is the size of the unit cell?

Group C

Long-Answer Type Questions

Answer any three questions

7. (a) Derive an expression for the velocity of a forced oscillator. Discuss the variation of velocity amplitude with driving force frequency and show its behaviour graphically.
   (b) Define temporal coherence and spatial coherence?
8. (a) What is missing order?
   (b) what you mean by resolving power of an optical instrument?
   (c) A parallel beam of light of wavelength 5890 Å falls normally on a plane transmission grating having 4250 lines/cm. Find the angle of diffraction for maximum intensity in first order.
   (d) State Brewster's law and hence prove that the angle between the reflected and refracted ray is 90°.
9. (a) Define (i) unit cell (ii) space lattice (iii) basis
   (b) Define atomic packing factor. Calculate it for BCC and FCC structure.
   (c) What is white x-ray?
   (d) Show that a free electron at rest cannot absorb a photon.
   (e) Imagine an electron to be somewhere in the nucleus whose dimension is 10^{-14} m. What is the uncertainty in momentum?
10. (a) What are the difference between ordinary light and laser light?
    (b) What is optical resonator? Discuss it in brief.
    (c) Discuss the operation of a ruby laser system with the help of energy diagram.
    (d) What is hologram? Explain the process of recording and reconstruction of hologram.

Solved WBUT Questions of 2005

OSCILLATIONS (SHM, FDV and FV)

Group A

Multiple-Choice Type Questions

1. (i) On superimposing two mutually perpendicular simple harmonic motions, we get circular Lissajous figures when their phase difference (ϕ) and amplitudes (P, Q) are as follows:
   (a) $\phi = 0, P = Q$
   (b) $\phi = \pi/2, P = Q$
   (c) $\phi = 0, P \neq Q$

Ans. (b)

Group B

Long-Answer Type Questions

1. (a) Define damped vibration.
   (b) Write down the differential equation for damped vibratory motion explaining the physical significance of each term in equation.
   (c) Obtain solution for the damped oscillatory motion.

Ans. (a) When the amplitude of vibration of a vibrating body diminishes with time due to the presence of some external resistive force, the vibration of such a body is known as damped vibration.

(b) The differential equation for a damped vibratory motion is given by

$$m\frac{d^2y}{dt} + \beta\frac{dy}{dt} + ky = 0$$

where m is the mass of the vibrating body, β is the resistive force which is proportional to the velocity of the vibrating body and k is the restoring force per unit displacement.

(c) Refer to Article 2.4 of Chapter 2.

OPTICS (Interference, Diffraction and Polarization)

Group A

Multiple-Choice Type Questions

1. (i) Polarisation of light establishes that light is essentially

(a) corpuscular matter (b) longitudinal wave (c) transverse wave

Ans. (c)

(ii) The central fringe in a Newton's ring experiment with a monochromatic light is

(a) bright (b) dark (c) white

Ans. (b)

(iii) The intensity of the principal maxima in the spectrum of a grating with N number of lines is proportional to

(a) $\frac{1}{N}$ (b) N (c) N^2

Ans. (c)

Group B

Long-Answer Type Questions

1. (a) What is meant by polarisation of light?

Ans. Light is a transverse wave. The electric and magnetic vectors of light wave vibrate symmetrically in a plane perpendicular to the direction of propagation of light. So, the electric and magnetic vectors vibrate in all planes which contain the direction of propagation. Polarisation is a process by which one can confine the vibration of the electric and magnetic vectors in a single plane.

(b) State Brewster's law and hence prove that the angle between the reflected and the refracted rays is 90°.

Ans. Refer to Article 6.7 of Chapter 6.

(c) The refractive index of glass plate is 1.6. Calculate the polarisation angle and the corresponding angle of the refracted ray.

Ans. If θ_p be the angle of polarisation, then

$$\mu = \tan\theta_p$$

or, $$\theta_p = \tan^{-1}(\mu) = \tan^{-1}(1.6)$$

$\therefore$ $$\theta_p = 57.99°$$

If θ_r be the corresponding angle of refraction, then

$$\theta_p + \theta_r = 90°$$

$\therefore$ $$\theta_r = 90° - 57.99°$$

or, $$\theta_r = 32.01°$$

2. (a) What are coherent sources?
   (b) Show that the laws of conservation of energy is not violated in the interferance process.
   (c) In a Young's double silt experiment the distance between the two coherent sources is 1.15 mm. Calculate the fringe width that would be observed on a screen placed at a distance of 85 cm from the source. The wavelength of light used is 5893 Å.

Ans. (a) Coherence is the predictable correlated motion of two waves when there is a fixed amplitude and phase relationship between them. If two light waves coming out of two sources are coherent, then these sources are also said to be coherent.
   (b) Refer to Article 4.8 of Chapter 4.
   (c) The fringe width is given by

$$\beta = \frac{D\lambda}{2d}$$

Here, the distance between two sources is $d = 1.15$ mm $= 0.115$ cm.

The distance between source and screen $D = 85$ cm.

Wavelength of light $\lambda = 589.3$ nm

$$= 589.3 \times 10^{-7} \text{ cm}$$

$$\therefore \quad \beta = \frac{5.893 \times 10^{-5} \times 85}{2 \times 0.115} \text{ cm}$$

or, $\beta = 2177.85 \times 10^{-5}$ cm

or, $\beta = 0.22$ mm

3. (a) Distinguish between Fraunhoffer and Fresnel diffraction.
   (b) Obtain an expression for the resultant intensity in a single slit Fraunhoffer diffraction process and show the intensity pattern graphically.

Ans. (a) Refer to Article 5.3 of Chapter 5.
   (b) Refer to Article 5.4 of Chapter 5.

4. Write short notes as the following:
   (a) Fresnel's biprism
   (b) Nicol prism as polarizer and analyzer.

Ans. (a) Out of syllabus
   (b) Refer to Article 6.12.2 of Chapter 6.

5. (a) What is the difference between the extraordinary and the ordinary ray propagating through an uniaxial crystal.
   (b) Define resolving power of a grating.

Ans. (a) When a light ray passes through a crystal it generates two images. This happens because while light passes through the crystal it gets split into two rays called ordinary ray and extraordinary ray. The ordinary ray obeys the laws of refraction but the extraordinary ray does not obey.
   (b) The resolving power of a plane grating is its ability to just distinguish (resolve) two nearby spectral lines with two close wavelengths.

LASER

Group A

Multiple-Choice Type Question

1. (i) For laser action to occur, the medium used must have at least:

(a) 4 energy levels (b) 2 energy levels (c) 3 energy levels

Ans. (c)

Group B

Long-Answer Type Questions

1. State the characteristics of the laser beam.

Ans. Refer to Article 7.2 of Chapter 7.

2. (a) What is population inversion? Explain?

(b) How higher probability of stimulated emission compared to that of spontaneous emission is achieved in laser?

Ans. (a) Refer to Article 7.6 of Chapter 7.

(b) Refer to Article 7.5 of Chapter 7.

QUANTUM PHYSICS

Group A

Multiple-Choice Type Questions

1. (i) The wavelength at which the spectral energy density of emitted radiation at temperature T from a black body attains maximum value is proportional to

(a) $\frac{1}{T}$ (b) T (c) T^4 (d) $T^{3/2}$

Ans. (a)

(ii) The de Broglie wavelength of a body of mass m and kinetic energy E is

(a) $\frac{2mh}{\sqrt{E}}$ (b) $\frac{h}{\sqrt{2m\,E}}$ (c) $\frac{h}{2m\,E}$ (d) $\frac{\sqrt{2mh}}{E}$

Ans. (b)

(iii) Total probability of finding a moving particle represented by a wave function $\psi(\bar{r})$ in the whole space is

(a) Zero (b) ∞ (c) 1 (d) $|\psi(\bar{r})|^2$

Ans. (c)

(iv) A coin and a six–faced dice are thrown. The probability that the coin shows tail and the dice shows five is

(a) $\frac{7}{12}$ (b) $\frac{1}{8}$ (c) $\frac{1}{12}$ (d) $\frac{1}{6}$

Ans. (c)

Group (B)

Long-Answer Type Questions

1. What is Heisenberg's uncertainty principle?

Ans. Refer to Article 9.6.1 of Chapter 9.

2. (a) Derive an exprenion for the Compton shift in wavelength for a photon scattered from a free electron, at an angle θ.
   (b) At which angle the shift will be maximum?
   (c) Calculate the compton wavelength (in Å) for an electron.

Ans. (a) Refer to Article 9.4.1 of Chapter 9.
(b) When the angle of scattering (θ) is π.
(c) The compton wavelength for an electron is given by

$$\lambda_c = \frac{h}{m_0 c} = 2.42 \times 10^{-3} \text{ mm.}$$
$$= 0.0242 \text{ Å}$$

3. (a) What is the experimental evidence in favour of de Broglie's hypothesis of matter waves?
   (b) Why in case of moving electrons quantum mechanics is used while for moving cars we use newtonian mechanics? Explain.

Ans. (a) Refer to Article 9.5.8 of Chapter 9.
(b) In case of moving electron we use quantum theory because its velocity is comparable to that of light in free space. But in case of a car $v << c$ where $c = 3 \times 10^8$ m/s

4. Write short note on Planck's radiation law.

Ans. Refer to Article 9.2.5 of Chapter 9.

CRYSTALLOGRAPHY

Group A

Multiple-Choice Type Questions

1. (i) [111] plane of cubic crystal is (a) perpendicular to x-axis, (b) perpendicular to y-axis, (c) perpendicular to z-axis, (d) none of these.

Ans. (d)

(ii) If λ_L and λ_K are the wavelengths of L and K x-rays respectively, then (a) $\lambda_L > \lambda_K$, (b) $\lambda_L < \lambda_K$ (c) $\lambda_L = \lambda_K$, (d) $\lambda_K = 4\,\lambda_L$

Ans. (a)

Group B

Long-Answer Type Questions

1. What is Bravais lattice?

Ans. Refer to Article 10.2.3 of Chapter 10.

2. (a) Show that in a cubic crystal of side 'a' the spacing between consecutive parallel planes of Miller indices (hkl) is

$$d_{hkl} = \frac{a}{\sqrt{h^2 + K^2 + l^2}}$$

(b) Copper has f.c.c. structure and the atomic radius is 0.1278 nm. Calculate its density and the inter planar spacing for (321) planes. Take the atomic wt. of copper as 63.5.

Ans. (a) Refer to Article 10.9 of Chapter 10.

(b) Refer to Example 10.4 of Chapter 10

3. (a) Distinguish between continuous and characteristic x-ray spectra.

(b) Explain and derive Bragg's law of x-ray diffraction from a crystal.

(c) The spacing of the planes in a crystal is 1.2 Å and the angle for the 1st order Bragg reflection is 30°. Determine the energy of the X-ray in eV.

Ans. (a) Refer to Article 10.13 of Chapter 10.

(b) Refer to Article 10.16 of Chapter 10.

(c) Refer to Example 10.13 of Chapter 10.

Solved WBUT Questions of 2006

OSCILLATIONS (SHM, FDV and FV)

Group A

Multiple-Choice Type Questions

1. (i) To get a circular Lissajfous figure, the phase difference (ϕ) and the amplitudes (a and b) of two superimposing mutually perpendicular simple harmonic motions respectively are

(a) $\phi = 0, a = b,$ (b) $\phi = \frac{\pi}{2}, a = b$ (c) $\phi = \frac{\pi}{2}, a \neq b,$ (d) $\phi = 0, a \neq b$

Ans. (b)

Group B

Long-Answer Type Questions

1. State the algebraic relation how the displacement is related to time in case of a damped harmonic motion. Derive the relation between the damping constant and logarithmic decrement. Derive

the relation between the first throw of a ballistic galvanometer with the amplitude of undamped oscillation in terms of the logarithmic decrement.

Ans. For the first part refer to Article 2.4 of Chapter 2.

In case of a ballistic galvanometer one requires to know the undamped throw whereas the practical throw is always to some extent damped. Thus the need arises in estimating undamped amplitude from the observed damped amplitude. Let the motion be started at time $t = 0$ without any initial displacement but with an initial velocity. The solution in this case is of the form $y = ce^{-kt}\sin(pt + \delta)$ where C is the amplitude in absence of damping. Now, with damping the first amplitude say C_1 is obtained at time $t = \frac{T}{4}$.

Thus $$C_1 = Ce^{-\frac{kT}{4}} = Ce^{-\lambda/2}$$

Hence $$C = C_1 e^{\lambda/2}$$

Usually $\lambda << 1$. So we can write

$$C = C_1\left(1 + \frac{\lambda}{2}\right) \qquad [\because \; e^x = 1 + x + \ldots]$$

Thus, measuring log decrement λ and noting first amplitude C_1 we can calculate the amplitude in absence of damping.

2. (a) Write down the differential equation of a forced vibration, explain each term.
(b) Starting from the solution of the equation for forced vibration, explain the phenomenon of amplitude and velocity resonance deriving the value of the driving frequency in each case.
(c) Starting from the equation of plane progressive wave in one dimension, derive the differential equation for the wave.

Ans. (a) Refer to Article 3.3 of Chapter 3.
(b) Refer to Articles 3.4 and 3.6 of Chapter 3.
(c) The equation of a plane progressive wave equation is represented as

$$y(x, t) = a\sin(\omega t - kx)$$

Differentiating the above equation twice with respect to t, we get

$$\frac{\partial^2 y}{\partial x^2} = -a\omega^2 \sin(\omega t - kx) = -w^2 y$$

Again differentiating twice with respect to x, we get,

$$\frac{\partial^2 y}{\partial x^2} = -ak^2 \sin(\omega t - kx) = -k^2 y$$

$$\therefore \quad \frac{\partial^2 y}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2}$$

or, $$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \quad \text{where} \quad v^2 = \frac{\omega^2}{k^2}$$

which is the required differential equation.

OPTICS (Interference, Diffraction and Polarization)

Group A

Multiple-Choice Type Questions

1. (i) Angle of diffraction of the first order maximum in a diffraction grating
   - (a) increases if the separation between the rulings is increased.
   - (b) decreases if the separation between the rulings is increased.
   - (c) does not change if the separation between the rulings is increased.
   - (d) The change depends on the wavelength of the incident light.

Ans. (b) or (d)

(ii) Two waves having intensities in the ratio of 9 : 1 produce interference. The ratio of maximum to minimum intensity is equal to

(a) 10 : 8 (b) 9 : 1 (c) 4 : 1 (d) 2 : 1

Ans. (c)

(iii) Newton's rings are example of interference fringes of
- (a) equal inclination, coherence is obtained by amplitude division.
- (b) equal thickness, coherence is obtained by amplitude division.
- (c) equal thickness, coherence is obtained by wave front division.
- (d) equal inclination, coherence is obtained by wave front division.

Ans. (b)

(iv) The plane of vibration and the plane of polarisation of a beam of plane polarised light
- (a) are identical to each other.
- (b) are orthogonal to each other.
- (c) make an angle, which depends on the colour of light.
- (d) rotate with respect to each other along the path of the beam.

Ans. (b)

Group B

Long-Answer Type Questions

1. (a) A polarizer and an analyzer are oriented so that the maximum intensity is achieved. To what fraction of its maximum value is the intensity reduced when the analyzer is rotated through 60°?

   (b) The intensity distribution for single slit diffraction is

$$I = I_0 (\sin^2 \beta/\beta^2)$$

where $$\beta = \pi b (\sin \theta)/\lambda$$

b is the width of the slit, and λ is the wavelength of light. Show that secondary maxima are given by the equation

$$\tan \beta = \beta$$

Ans. (a) Malus law is given by

$$I_\theta = I_i \cos^2 \theta$$

where I_i is the intensity of the light beam incident on the polarizer, I_θ is the intensity of the output light beam and θ is the angle between the planes of transmission of the polarizer and the analyzer.

In this case, I_θ is maximum, so $\theta = 0$.

$$\therefore \quad I_0 = I_i \quad \text{...(i)}$$

when $\theta = 60°$, $I_{60} = I_i \cos 60° = \frac{1}{2} I_i$

$$\therefore \quad \frac{I_{60}}{I_0} = \frac{\frac{1}{2} I_i}{I_i} = \frac{1}{2}$$

(b) Refer to Article 5.4 of Chapter 5.

2. (a) Explain the difference between interference and diffraction.
   (b) Derive the expression of intencity at a point for Fraunhofer diffraction due to a double slit. Draw the intensity distribution curve (diffraction pattern) and explain it.
   (c) A parallel beam of wavelength (589 nm) falls normally on a plane transmission grating having 4250 lines/cm. Find the angle of diffraction for maximum intensity in first order.

Ans. (a) Refer to Article 5.3 of Chapter 5.
(b) Refer to Article 5.5 of Chapter 5.
(c) For a transmission grating the wavelength λ of the incident light is given by

$$\lambda = \frac{\sin \theta_n}{mn}$$

Here, $\lambda = 589 \times 10^{-5}$ cm
$m = 4250$ lines/cm
$n = 1$ (1st order)

$$\therefore \quad \sin \theta_n = \lambda mn$$
$$\therefore \quad \sin \theta_1 = (5.80 \times 10^{-5})(4250) \times (1)$$
$$\text{or,} \quad \sin \theta_1 = 0.25$$
$$\therefore \quad \theta_1 = \sin^{-1}(0.25)$$
$$\therefore \quad \theta_1 = 14.5°$$

LASER

Group A

Multiple-Choice Type Questions

1. (i) The population inversion in preparing laser beam can be achieved
   (a) when one of the excited states is more populated than the ground state.

(b) when one of the excited states is less populated than the ground state.

(c) when the population of one excited state and the ground state are equal.

(d) irrespective of any relationship between the population in the excited and the ground states.

Ans. (a)

Group B

Long-Answer Type Questions

1. Describe briefly the principle of operation of laser.

Ans. Refer to Article 7.5 of Chapter 7.

QUANTUM PHYSICS

Group A

Multiple-Choice Type Questions

1. (i) An α-particle is 4 times heavier than a proton. If a proton and an α-particle are moving with the same velocity, how do their de Broglie wave lengths compare?

Ans. (b)

(ii) Uncertainty principle tells that

(a) A particle can have only position but no momentum.

(b) A particle can have only momentum but no position.

(c) One cannot determine simultaneously the position and momentum of a particle.

(d) One cannot determine simultaneously. The position and momentum of a particle.

Ans. (d)

(iii) Emissive power of a black body kept at an absolute temperature T is proportional to

(a) T^3 (b) T^4 (c) T^5 (d) T^{-1}

Ans. (b)

Group B

Long-Answer Type Questions

1. What is the ratio of Compton shift for scattering at 0° and 90°?

Ans. The Compton shift is given by,

$$\Delta\lambda = \frac{h}{m_0 c}(1 - \cos\theta) = \frac{2h}{m_0 c}\sin^2\frac{\theta}{2}$$

Now, for $\theta = 90°$, $\Delta\lambda_{90} = 2\left(\frac{h}{m_0 c}\right) \times \left(\frac{1}{\sqrt{2}}\right)^2$

$\therefore \quad \Delta\lambda_{90^\circ} = \dfrac{6.62 \times 10^{-34}}{9.1 \times 10^{-31} \times 3 \times 10^8} = 0.0245\text{A}^\circ$

For $\theta = 0°, \Delta\lambda_0 = 0$ Å

The required ratio is given by

$$r = \frac{\Delta\lambda_0}{\Delta\lambda_{90}} = 0$$

2. State clearly, explaining all the terms, the planck's law, Rayleighjeans law and Wein's displacement law for radiation. Find out the two limits at which the planck's formula reduces to the other two.

Ans. Planck's radiations law can be written as follows:

$$u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{\exp(hc/\lambda kT) - 1} d\lambda$$

where $u_\lambda d_\lambda$ represents the energy density in the wavelength range between λ and $\lambda + d\lambda$, c represents the velocity of light in the vacuum and h is Planck's constant.

Rayleigh – Jeans law can be expressed

as $$u_\lambda d\lambda = \frac{8\pi}{\lambda^4} kT\, d\lambda$$

And wein's displacement law can be expressed as follows:

$$\lambda_{max} T = \text{constant}$$

where λ_{max} represents that wavelength at which maximum radiation is obtained and T is the absolute temperature.

Derivation of Rayleigh – Jeans law

Planck's formula is given by

$$u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{\exp\left(\dfrac{hc}{\lambda kT}\right) - 1} d\lambda$$

where $\lambda \to \infty, \dfrac{hc}{\lambda KT} \to 0$

So, Planck's formula can be written as

$$u_\lambda d\lambda = \frac{8\ \pi hc}{\lambda^5} \frac{1}{1 + \dfrac{hc}{\lambda kT}\left(\dfrac{hc}{\lambda kT}\right)^2 + 1} d\lambda$$

Now, we an neglect the higher powers of $\left(\dfrac{hc}{\lambda kT}\right)$ as $\dfrac{hc}{\lambda kT} \to 0.$

So, the above equation reduces to

$$u_\lambda d_\lambda = \frac{8\ \pi}{\lambda^4} kT\, d_\lambda$$

or, $$u_\lambda = \frac{8\ \pi}{\lambda^4} kT$$

or, $$E_\lambda = \frac{c}{4} u_\lambda = \frac{c}{4} \times \frac{8\pi}{\lambda^4} kT$$

$\therefore \qquad E_\lambda = 2\frac{\pi c}{\lambda^4}kT$

This is Rayleigh – Jeans formula. For derivation of Wein's law refer to Article 9.2.6 (a).

3. Dercribe briefly the Davisson–Germer experiment. What inferences may you draw from this experiment? Explain what you mean by "wave packet".

Ans. Refer to Article 9.5.8 and Article 9.5.2 of Chapter 9.

CRYSTALLOGRAPHY

Group A

Multiple-Choice Type Questions

1. (i) Miller indices of a plane which have intercepts of 2, 3 and 4 units along the three axes are

(a) (6, 4, 3) (b) (2, 3, 4) (c) (3, 2, 1) (d) (2, 3, 2)

Ans. (a)

(ii) Which of following wavelengths falls in the X-ray range?

(a) 1 mm (b) 100 mm (c) 0.001 mm (d) 1000 mm

Ans. (a)

(iii) When two parallel X-rays of wavelength λ, are incident at an angle θ on a crystal with lattice separation d, constructive interference would be observed when (n is an integer).

(a) $n\lambda = 2d \sin\theta$, (b) $n\lambda = d\sin\theta$ (c) $n\lambda = d\sin 2\theta$ (d) $n\lambda = 2d\sin 2\theta$

Ans. (a)

Group B

Long-Answer Type Questions

1. The distance between (100) planes in a body–centred cubic structure is 0.232 nm. What is the size of the unit cell? What is the radius of the atom?

Ans. Refer to Example 10.3 of Chapter 10.

Solved WBUT Questions of 2007

OSCILLATIONS (SHM, FDV and FV)

Group A

Multiple-Choice Type Questions

1. (i) Two mutually perpendicular oscillations with same frequency, amplitude but phase difference δ will produce closed curve with non-zero area enclosed

(a) for all values of δ except $\delta = 0$.

(b) only for $\delta = \frac{\pi}{2}$.

(c) for all values of δ except $\delta = 0$ and $\delta = \pi$.

(d) for all values of $\delta \geq \frac{\pi}{2}$.

Ans. (b)

(ii) Waves originating from a point source and travelling in an isotropic medium is described as

(a) $\phi = \phi_0 \exp\,[i(kr - \omega t)]$

(b) $\phi = \phi_0 \exp\left[\frac{i(kr - \omega t)}{r}\right]$

(c) $\phi = \phi_0 \exp\left[\frac{i(kr - \omega t)}{r^2}\right]$

(d) $\phi = \phi_0 \exp\left[\frac{i(kr + \omega t)}{r}\right]$

Ans. (b)

(iii) Example of weakly damped harmonic oscillator is

(a) dead-bead galvanometer

(b) tangent galvanometer

(c) ballistic galvanometer

(d) discharge of a charged capacitor through a resistance.

Ans. (c)

Group B

Long-Answer Type Questions

1. If a harmonic oscillator of mass m and natural frequency ω_0 is driven by a force $F_0 \sin \omega t$ and the damping is proportional to $2\,\mu$ times the velocity of the oscillator then the displacement is given by

$$x = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\,\mu^2\,\omega^2}} \sin\,(\omega t - \theta)$$

where $\tan\theta = \dfrac{2\,\mu\omega}{(\omega_0^2 - \omega^2)}$

Using the above relation (you need not deduce the relation), show that at velocity resonance the velocity is in phase with the driving force.

Ans. The displacement is given by

$$x = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\,\mu^2\,\omega^2}} \sin\,(\omega t - \theta)$$

or, $\quad x = A \sin\,(\omega t - \theta) \quad$ (say)

where $\tan\theta = \dfrac{2\,\mu\omega}{(\omega_0^2 - \omega^2)}$

Here, θ is the phase difference of the displacement x (at time t) with the driving force.

$\therefore$ the velocity of the oscillator is given by

$$v = \frac{dx}{dt} = A\omega \cos\,(\omega t - \theta)$$

or, $$v = (A\omega) \sin \left\{ \omega t + \left(\frac{\pi}{2} - \theta \right) \right\} \quad ...(1)$$

Given, $$\tan \theta = \frac{2 \mu \omega}{(\omega_0^2 - \omega^2)} \quad ...(2)$$

Let us now observe the variation of the phase θ of the oscillator with respect to the angular frequency ω of the driving force $F_0 \sin (\omega t)$. If $\omega_0 = \omega$ (which is the case at velocity resonance), then from (2), we get

$$\tan \theta = \alpha = \tan \frac{\pi}{2}$$

Hence, $$\theta = \frac{\pi}{2}$$

Putting this value of θ in Eq. (1), we get

$$v = A\omega \sin \left(wt + \frac{\pi}{2} - \frac{\pi}{2} \right)$$

or, $$v = A \sin (\omega t) \quad ...(3)$$

The driving force F is given by

$$F = F_0 \sin (\omega t) \quad ...(4)$$

Now, comparing Eqs. (3) and (4), we can conclude that at velocity resonance, the velocity is in phase with the driving force.

2. If $y = f_1 (x - vt) + f_2 (x + vt)$ where f_1 and f_2 are two functions then show that y satisfies the wave equation.

Ans. $$y = f_1(x - vt) + f_2(x + vt) \quad ...(1)$$

Differentiating with respect to x, we get

$$\frac{\partial y}{\partial x} = f_1' (x - vt) + f_2' (x + vt)$$

or, $$\frac{\partial^2 y}{\partial x^2} = f_1'' (x - vt) + f_2'' (x + vt) \quad ...(2)$$

Now, differentiating with respect to t, we get

$$\frac{\partial y}{\partial t} = - v f_1' (x - vt) + v f_2' (x + vt)$$

or, $$\frac{\partial^2 y}{\partial t^2} = v^2 f_1'' (x - vt) + v^2 f_2''(x + vt)$$

or $$\frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} = f_1''(x - vt) + f_2''(x + vt) \quad ...(3)$$

Now, comparing Eqs. (2) and (3), we get

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

which is the wave equation.

$\therefore$ $y = f_1 (x - vt) + f_2 (x + vt)$ satisfies the wave equation.

3. (a) Calculate the time period of the liquid column of length l in a U-tube, if it is depressed in one arm by x, d is the density of liquid and A is the cross-sectional area of each arm of the U-tube.

(b) In damped harmonic motion, calculate the time in which the energy of the system falls to $(1/e)$ times the initial value.

(c) Write down the differential equation of a series L-C-R circuit driven by a sinusoidal voltage. Identify the natural frequency of this circuit. Find out the condition that this circuit will show an oscillatory decay and find out the relaxion time.

(d) Find out the value of the driving frequency at which the voltage across the capacitor is maximum (you may use the expression provided in question No. 1).

Ans. (a) Refer to Example 1.6 of Chapter 1.

The total energy of a particle undergoing damped oscillation is given by

$$E = E_k + E_p$$

where E_k is the kinetic energy and E_p is the potential energy

$$\therefore \quad E = E_k + E_p = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \quad ...(1)$$

The general solution of the differential equation of a damped oscillator is given by,

$$y = Ae^{-bt}\sin(\omega't + \delta) \quad ...(2)$$

where $\omega' = \sqrt{\omega^2 - b^2}$, ω being the natural angular frequency.

$\therefore$ the velocity is given by

$$v = \frac{dy}{dt}$$

or,
$$v = A\omega' e^{-bt}\cos(\omega't + \delta) - Abe^{-bt}\sin(\omega't + \delta) \quad ...(3)$$

If the body is suddenly excited by giving a velocity v_0 while it is in the mean position, i.e., at $t = 0$, then

$$\left.\frac{dy}{dt}\right|_{t=0} = v_0 \quad \text{and} \quad \delta = 0$$

So, the Eq. (3) reduces to

$$v_0 = A\omega' \quad \text{or} \quad A = \frac{v_0}{\omega'}$$

$$\therefore \quad y = \frac{v_0}{\omega'}e^{-bt}\sin(\omega't) \quad ...(4)$$

and
$$v = \frac{dy}{dt} = \frac{v}{\omega'}\omega' e^{-bt}\cos(\omega't) - \frac{v_0}{\omega'}be^{-bt}\sin(\omega't)$$

or,
$$v = v_0 e^{-bt}\cos(\omega't) \quad [\because \; b \text{ is very small}] \quad ...(5)$$

$$\therefore \quad E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

or,
$$E = \frac{1}{2}mv_0^2 e^{-2bt}\cos^2(\omega't) + \frac{1}{2}m\omega^2\frac{v_0^2}{\omega'^2}e^{-2bt}\sin^2(\omega't) \quad [\because \; k = m\omega^2] \quad ...(6)$$

In case of damping, $\omega' = \sqrt{\omega^2 - b^2} \approx \omega$ $[\because \; b$ is very small$]$

$\therefore \quad E = \frac{1}{2} m v_0^2 e^{-2bt} \cos^2 \omega t + \frac{1}{2} m v_0^2 e^{-2bt} \sin^2 \omega t$

or, $\quad E = \frac{1}{2} m v_0^2 e^{-2bt} (\cos^2 \omega t + \sin^2 \omega t)$

$\therefore \quad E = \frac{1}{2} m v^2 e^{-2bt}$

when $\quad t = 0, E = \frac{1}{2} m v^2 = E_0$ (say)

$\therefore \quad E = E_0 e^{-2bt}$

Now, if $t = \frac{1}{2b} = \tau$ (say), then $E = \frac{E_0}{e}$

$\therefore$ The time during which the value of the energy (E) becomes $\frac{1}{e}$ of its initial value is given by

$$\tau = \frac{1}{2b}$$

(c) An LCR circuit driven by a sinusoidal voltage is shown in the Fig. 1.

L is the inductance, C is the capacitance and R is the resistance, $V (= V_0 \sin \omega' t)$ is the sinusoidal driving voltage. The circuit at voltage equation is given by

$$L \frac{di}{dt} + Ri + \frac{q}{c} = V_0 \sin (\omega' t)$$

or, $$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{i}{c} = \omega' V_0 \cos (\omega' t) \quad ...(1)$$

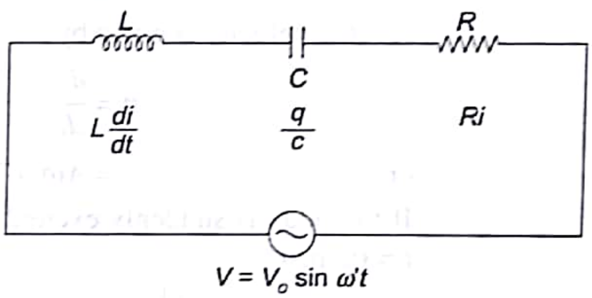


Fig. 1 An LCR circuit

The Eq. (1) is the required equation. We can rewrite this equation as follows:

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = \frac{\omega' V_o}{L} \cos (\omega' t) \quad ...(2)$$

Let us now compare Eq. (2) Eq. (3.1) of Ch. 3 which is given by

$$\frac{d^2 y}{dt^2} + 2b \frac{dy}{dt} + \omega^2 y = f \cos (\omega' t) \quad ...(3)$$

This equation is the equation for a damped (mechanical) oscillator.

If we write $i = y$, then we can write,

$$\frac{R}{L} = 2b, \ \omega^2 = \frac{1}{LC} \quad \text{and} \quad \frac{\omega' V_0}{L} = f$$

$\therefore$ The natural frequency of the LCR circuit is given by

$$\nu = \frac{1}{2\pi}\frac{1}{\sqrt{LC}} \quad [\because \quad \omega = 2\pi\nu]$$

For showing oscillatory motion, b must be small enough, so the resistance should be small and the inductance should be high.

The relaxation time is given by

$$T_r = \frac{1}{b} = \frac{1}{R/2L} = \frac{2L}{R}$$

(d) In the electric circuit the current i in complex form is given by

$$i = \frac{E_0 e^{j\omega t}}{R + j\left(\omega L - \frac{1}{\omega c}\right)}$$

where $Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$ is called complex impedance and $X = \left(\omega L - \frac{1}{\omega C}\right)$ is called reactance.

The voltage across the capacitor is given by

$$V_c = iR_c = \frac{i}{C\omega} \quad \text{where} \quad R_c = \frac{1}{\omega C}$$

or, $$V_c = \frac{E_0 e^{i\omega}/(\omega C)}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$\therefore$ $$|V_c| = \frac{E_0/(\omega C)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

or $$|V_c| = \frac{E_0}{\omega C\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

$$|V_c| = \frac{E_0}{\sqrt{\omega^2 C^2 R^2 + (\omega^2 LC - 1)^2}}$$

In order to find the maximum value let us differentiate the denominator of $|V_c|$ with respect to ω

$\therefore$ $$\frac{d}{d\omega}[\omega^2 C^2 R^2 + (\omega^2 LC - 1)^2] = 0$$

or, $$2\omega R^2 C^2 + 2(\omega^2 LC - 1) \times 2\omega LC = 0$$

or, $$2\omega R^2 C^2 + 4\omega LC(\omega^2 LC - 1) = 0$$

or, $$2LC(\omega^2 LC - 1) + R^2 C^2 = 0$$

or, $$2\omega^2 L^2 C^2 - 2LC + R^2 C^2 = 0$$

or, $$\omega^2 = \frac{-R^2C^2 + 2LC}{2L^2C^2} = \frac{1}{LC}\left(1 - \frac{R^2C}{2L}\right)$$

or, $$\omega = \left\{\frac{1}{LC}\left(1 - \frac{R^2C}{2L}\right)\right\}^{\frac{1}{2}}$$

which is the required driving frequency.

LASER

Group A

Multiple-Choice Type Questions

1 (i) In a ruby laser, population inversion is achieved by

(a) optical pumping (b) inelastic atom-atom collision
(c) chemichal reactions (d) applying strong electric field.

Ans. (a)

Group B

Long-Answer Type Questions

1. (a) Explain working principle of He-Ne laser with energy level diagram.

Ans. (a) Refer to Article 7.10.2 of Chapter 7.

QUANTUM PHYSICS

Group A

Multiple-Choice Type Questions

1. (i) Davisson and Germer studied electron diffraction with nickel crystal and found a first order peak at 65° with electron beam of 54 eV. If instead a 216 eV beam were used then the peak would have bean at

(a) 27° (b) 54° (c) 130° (d) 260°

Ans. (a)

(ii) If visible light is used to study the Compton scattering then the compton shift will be

(a) negative (wave length of the scattered light will be laser).
(b) more positive than what is observed with X-ray.
(c) zero
(d) positive but not detectable in the visible window.

Ans. (d)

Long-Answer Type Questions

1. Show graphically how the energy density versus frequency plot of blackbody radiation is changed if the temperature is increased.

Ans. Refer to figure 9.2 of Chapter 9.

2. Starting from de Broglie's hypothesis show that the group velocity associated with a particle is same as the particle velocity.

Ans. Refer to Article 9.5 of Chapter 9.

3. Assume that an electron is inside a nucleus of radius 10^{-15} m. Calculate from the uncertainty principle the maximum kinetic energy of the electron. [Given $h = 6.63 \times 10^{-34}$ J-s, $m_e = 9.1 \times 10^{-31}$ kg]

Ans. Refer to Article 9.8, Problem 9.27.

CRYSTALLOGRAPHY

Group A

Multiple-Choice Type Questions

1. (i) Assuming that the atoms in a crystal are spheres of equal size and touching each other, it can be shown that atomic radius of *bcc* is equal to

(a) $\frac{a}{2}$ (b) $\frac{\sqrt{3}a}{4}$ (c) $\frac{\sqrt{3}a}{2}$ (d) $\frac{a}{2\sqrt{2}}$

Ans. (b)

(ii) Origin of continuous X-ray is due to the process of

(a) ionization (b) inner orbital transition
(c) bremsstrahling (d) none of these

Ans. (c)

Long-Answer Type Questions

1. What is meant by Miller indices? Explain with example.

Ans. Refer to Article 10.8 of Chapter 10.

2. Calculate the interplanar spacing '*d*' of planes (1, 1, 1) in a simple cubic lattice of side *a*. Deduce the formula that you use.

Ans. Refer to Article 10.9 of Chapter 10

For (1, 1, 1) planes $h = k = l = 1$

$$\therefore \quad d = \frac{a}{\sqrt{h^2 + k^2 + l^2}} = \frac{a}{\sqrt{3}}$$

3. A beam of X-rays of wave–length 0.842 Å is incident on a crystal at a glancing angle of 8°35′, when first order Bragg's reflection occurs. Calculate the glancing angle for third order reflection.

Ans. Refer to worked out problem 10.10 of Chapter 10.

Solved WBUT Questions of 2008

OSCILLATIONS (SHM, FDV and FV)

Group A

Multiple-Choice Type Questions

1. (i) Two mutually perpendicular SHMs with equal time periods but different amplitudes are superposed. If the phase difference between these oscillations is 45°, them they form a

   (a) circle (b) straight line (c) ellipse (d) parabola

Ans. (c)

(ii) Motion of a system in critical damped condition is

   (a) oscillatory (b) damped oscillatory
   (c) harmonic (d) non-oscillatory

Ans. (d)

(iii) A plane wave travels in a medium in the positive X-direction. The displacements of the particles are given by

$$y(x, t) = 10 \sin 2\pi (t - 0.2\, x)$$

where x and y are measured in metre and t in seconds. The wavelength of the wave is

   (a) 0.2 m (b) 5 m (c) 2π m (d) 0.4 m

Ans. (b)

Group B

Long-Answer Type Questions

1. A particle is subjected to a harmonic restoring force and a damping. Its equation of motion is given by

$$m \frac{d^2x}{dt^2} = -sx - k\frac{dx}{dt}$$

Under the condition of critical damping, find the expression for displacement as a function of time. You can justify why this condition (critical damping) corresponds to the fastest non-oscillatory decay?

Ans. Refer to Article 2.4 of Chapter 2.

2. Establish the equation of a forced harmonic oscillator. Solve this equation to obtain the amplitude of steady state oscillations. What is Q-factor of a forced oscillator? The amplitude of an oscillator of frequency 200 cps falls to $\left(\frac{1}{10}\right)^{\text{th}}$ of its initial value after 2000 cycles, Calculate its relaxation time, quality factor, time in which its energy falls to $\left(\frac{1}{10}\right)^{\text{th}}$ of its initial value.

Ans. Refer to Article 3.3 and 3.7 of Chapter 3 and the solution of the problem is given below:

Here, frequency $\nu = 200\ cs^{-1}$

$$\text{Time period } T = \frac{1}{\nu} = \frac{1}{200} \text{ second}$$

Ratio of final amplitude A_f to initial amplitude A_i is given by

$$\frac{A_f}{A_i} = \frac{1}{10}$$

The time period to complete 2000 oscillations (t_1) is given by

$$t_1 = \frac{2000}{T} = \frac{2000}{200} = 10 \text{ seconds}$$

Initial amplitude, $A_i = Ae^{-bt}$

and the final amplitude $A_f = Ae^{-b}(t + t_1)$

$$\therefore \quad \frac{A_f}{A_i} = \frac{Ae^{-b(t+t_1)}}{Ae^{-bt}}$$

or, $$\frac{1}{10} = e^{-bt_1}$$

or, $$10^{-1} = e^{-bt_1}$$

or, $$-\log_e^{10} = -bt_1$$

or, $$bt_1 = \frac{1}{\log_{10}^e}$$

$$\therefore \quad b = \frac{1}{t_1 \log_{10}^e} = \frac{1}{10 \times 0.432}$$

or, $$b = 0.230$$

$\therefore$ The relaxation time τ_r is given by

$$\tau_r = \frac{1}{2b} = \frac{1}{0.46}$$

or, $$\tau_r = 2.174 \text{ second}$$

The quality factor Q is given by

$$Q = \frac{\omega}{2b} = \frac{2\pi\nu}{2b}$$

or, $$Q = \frac{2\pi \times 2000}{0.46} = 2730.4$$

Let t_2 be the time during which the energy falls to $\left(\frac{1}{10}\right)^{\text{th}}$ of its initial value,

$$\therefore \quad E_f = \frac{1}{10} E_i$$

Again $$E_f = E_i e^{-2bt_2}$$

or, $$\frac{E_i}{10} = E_i e^{-2bt_2}$$

or, $$10^{-1} = e^{-2bt_2}$$

or, $-1 = -2bt_2 \log_{10}^{e}$

or, $2bt_2 \log_{10}^{e} = 1$

or, $t_2 = \dfrac{1}{2b \times \log_{10}^{e}}$

or, $t_2 = \dfrac{1}{0.46 \times 0.432}$

or, $t_2 = 5.032$ second

OPTICS (Interference, Diffraction and Polarization)

Group A

Multiple-Choice Type Questions

1. (i) The electromagnetic wave is called transverse because
   (a) the electric field and the magnetic field are perpendicular to each other.
   (b) the electric field is perpendicular to the direction of propagation.
   (c) the magnetic field is perpendicular to the direction of propagation.
   (d) both the electric field and the magnetic field are perpendicular to the direction of propagation.

Ans. (d)

(ii) Optic axis is a direction along which
   (a) the extraordinary ray travels faster than the ordinary ray.
   (b) the ordinary ray travels faster than the extraordinary ray.
   (c) both the extraordinary ray and the ordinary ray travel with same velocity.
   (d) none of these.

Ans. (c)

(iii) A plane polarized light of intensity I_0 is incident on a polarizer whose axis of polarization is 30° with respect to the direction of propagation of the incident light. The intensity of light coming out of the polarizer is

   (a) I_0 (b) $\dfrac{\sqrt{3}I_0}{2}$ (c) $\dfrac{3I_0}{4}$ (d) $\dfrac{I_0}{4}$

Ans. (c)

(iv) The radii of the Newton's rings (R_n) depend on the wavelength of the incident light (λ) and the radius of curvature of the convex surface of the plano-convex lens (r) as follows:
   (a) R_n increases with λ and decreases with r.
   (b) R_n increases with r and decreases with λ.
   (c) R_n increases both with λ and r.
   (d) R_n decreases both with λ and r

Ans. (c)

(v) The relation between the path difference (Δl) and phase difference ($\Delta\phi$) is

(a) $\Delta\phi = \frac{2\pi}{\lambda}\Delta l$ (b) $\Delta\phi = \frac{\pi}{\lambda}\Delta l$

(c) $\Delta l = \frac{2\pi}{\lambda}\Delta\phi$ (d) $\Delta l = \frac{\pi}{\lambda}\Delta\phi$

Ans. (a)

(vi) In Fraunhoffer diffraction, the incident wave front is

(a) plane (b) spherical (c) cylindrical (d) of arbitrary shape

Ans. (a)

(vii) The separation between diffraction lines (Δ) for the pattern formed by a transmission grating depends on the number of rulings per inch (n) and the wavelength of the incident light (λ) as

(a) Δ increases with n but decreases with λ

(b) Δ decreases with n but decreases with λ

(c) Δ increases both with n and λ

(d) Δ decreases both with n and λ

Ans. (c)

(viii) Double slit interference pattern is the limiting case of double slit differaction pattern when the

(a) distance between the slit, tends to zero.

(b) distance of the source and the slit tend to infinity.

(c) slit widths tend to zero.

(d) distance between the slits tends to infinity

Ans. (c)

(ix) A light ray is passing through a calcite crystal. If the plane of vibration of the light ray is perpendicular to the optical axis the ray

(a) is an O-ray,

(b) is an E-ray

(c) has both non-zero O and E components

(d) cannot propagate through the crystal

Ans. (a)

(x) A source of light emits lights of frequencies between ν and $\nu + \Delta\nu$. Coherence time of the emergent light beam is T_c, then

(a) $T_c \propto \Delta\nu$ (b) $T_c \propto \frac{1}{\Delta\nu}$ (c) $T_c \propto \nu$ (d) $T_c \propto \frac{1}{\nu}$

Ans. (b)

(xi) The intensity of principal maxima in the spectrum of a grating with N number of lines is proportional to

(a) $\frac{1}{N}$ (b) N (c) N^2 (d) $\frac{1}{N^2}$

Ans. (c)

Group B

Long-Answer Type Questions

1. Two polarizers are placed at crossed position (angle between the polarizing planes are 90°), a third polarizer with angle θ with the first one is placed between them. An unpolarized light of intensity I is incident on the first one and passes through all three polarizers. Find the intensity of light that comes out.

Ans. Refer to worked out problems of Chapter 6, Problem 6.3.

2. Calculate the polarizing angle for light travelling from water of refractive index 1.33 to glass of refractive index 1.53.

Ans. (a) Refer to the worked out problems of Chapter 6, Problem 6.2.

3. (a) Explain briefly the action of a dichroic polaroid.
   (b) In Young's double slit interference distance between the coherent sources are 1.15 mm. Calculate the fringe width that would be observed on a screen placed at a distance of 85 cm from the source. The wavelength of light used is 5.893 Å.
   (c) Interference phenomenon does not violate the principle of conservation of energy. Justify it.

Ans. (a) Refer to Article 6.15 of Chapter 6.
(b) Refer to worked out problem of Chapter 4, Problem 4.6.
(c) Refer to Article 4.8 of Chapter 4.

4. A film of oil of refractive index 1.70 is placed between a plane glass plate and an equi-convex lens. The focal length of the lens is 1 metre. Determine the radius of the 10th dark ring when light of wavelength is 6000 Å.

Ans. Refer to worked out problems of Chapter 4, Problem 4.19.

5. A convex lens of focal length 40 is employed to focus the Fraunhofer diffraction pattern of a single slit of 0.3 mm width. Calculate the linear distance of the first order dark band from the central band. The wavelength of the light is 589 nm.

Ans. Refer to worked out problem of Chapter 5, Problem 5.3.

6. (a) Calculate the least width that a grating must have to resolve two components of the sodium-D line in the second order, the grating having 800 lines/cm. The wavelength for D_1 and D_2 lines of sodium are 5893 Å and 5896 Å respectively.
   (b) Deduce the missing order for a double slit Fraunhofer diffraction pattern if the slit width are 0.16 mm and they are 0.8 mm apart.
   (c) Two stars situated at a distance of 9.5×10^{12} km from a telescope of diameter 20 cm are sending light of wavelength 600 nm. Find the distance of the separation of the stars for which, they are just resolved.

Ans. (a) Refer to worked problem of Chapter 5, Problem 5.14.
(b) Refer to worked out problem of Chapter 5, Problem 5.5.
(c) The diameter of the telescope

$$D = 1.22\frac{\lambda L}{y}$$

Here, y = Distance of separation of the stars;

L = Distance of the telescope from stars = 9.5×10^{12} km = 9.5×10^{15} m;

λ = wave length of the light = 600 mm = 600×10^{-9} m

$$\therefore \quad y = 1.22 \times \frac{\lambda L}{D} = \frac{1.22 \times 600 \times 10^{-9} \times 9.5 \times 10^{15}}{20 \times 10^{-2}}$$

or, $$y = 3.477 \times 10^7 \text{ km}$$

7. Sodium light of wavelength 589 nm and 589.6 nm are made incident normally on a grating having 500 lines/cm. Calculate the angular dispersion of these lines in the spectrum of first order. By a simple calculation can you justify whether these lines are resolved or not?

Ans. Here wavelengths of the given rays are

$$\lambda_1 = 589.0 \text{ nm}$$
$$\lambda_2 = 589.6 \text{ nm}$$

Number of rulings/cm in the given grating $N = 500$ lines/cm

We have, $$\lambda = \frac{\sin \theta_n}{Nn} \quad \text{where } n \text{ is the order}$$

Let $$\theta_n = \theta'_n, \quad \text{and} \quad n = 1$$

For $\lambda = \lambda_1$, so we get

$$\lambda_1 = \frac{\sin \theta'_1}{N.(1)} \Rightarrow 589 \times 10^{-7} = \frac{\sin \theta'_1}{500 \times 1}$$

$$\sin \theta'_1 = 0.02945 = \sin 1.688°$$

$$\therefore \quad \theta'_1 = 1.688°$$

For $\lambda = \lambda_2$. Let $\theta_n = \theta''_1, n = 1$

$$\therefore \quad \lambda_2 = \frac{\sin \theta''_1}{N.(1)}$$

$$\Rightarrow \quad \sin \theta''_1 = N \times 2 = 500 \times 589.6 \times 10^{-7}$$

or $$\sin \theta''_1 = \sin 1.689°$$

$$\therefore \quad \theta''_1 = 1.689°$$

$\therefore$ angular dispersion θ_d is given by

$$\theta_d = \theta''_1 - \theta'_1 = 1.689° - 1.688°$$

$$\therefore \quad \theta_d = 0.001°$$

Let Nn be the number of ruling per cm just to resolve nth order spectrum

$\therefore$ resolving power $\quad R_p = n\,N_n$

To resolve 1st order the expression reduces to

$$R_p = (1)\,N_1$$

Again, resolving power is given by

$$R_p = \frac{\lambda}{\Delta\lambda}$$

$\therefore \quad N_1 = R_p = \dfrac{\lambda}{\Delta\lambda}$

or, $\quad N_1 = \dfrac{5893}{6} \quad \left[\because \; \lambda = \dfrac{\lambda_1 + \lambda_2}{2}\right]$

$\therefore \quad N_1 = 982.1$ rulings/cm

The given grating has $N = 500$ rulings/cm

$\therefore \quad N < N_1$

$\therefore$ the lines 589.0 nm and 589.6 nm are not resolved here.

8. The width of each slit of a double slit is 0.15 mm and they are separated by a distance of 0.45 mm. If the double slit produces Fraunhofer diffraction, find the angular position of the first minima and the missing orders.

Ans. Missing orders are obtained when interference maxima and diffraction minima correspond to the same value of direction angle θ, i.e.,

$$(a + b) \sin \theta_n = n\lambda \quad \text{[interference maxima]}$$

and $$a \sin \theta_m = m\lambda \quad \text{[diffraction minima]}$$

$$\therefore \quad \frac{a+b}{a} = \frac{n}{m} \quad [\because \text{ in this case } \theta_n = \theta_m]$$

Here $\quad a = 0.15$ mm $\quad$ and $\quad b = = 0.45$ mm

$$\therefore \quad \frac{n}{m} = \frac{0.15 + 0.45}{0.15} = \frac{0.60}{0.15} = 4$$

for values of $m = 1, 2, 3$ etc. and

$$n = 4, 8, 12, \text{ etc.}$$

So the 4th, 8th, 12th etc. orders of interference maxima will be missing in the diffraction spectrum.

9. (a) Find the intensity distribution for the Fraunhoffer diffraction pattern through a single slit.
   (b) Hence find the position of maxima and minima.

Ans. (a) Refer to Article 5.4 of Chapter 5.
Refer to Article 5.4 of Chapter 5.

10. A parallel beam of light of wave length 500 nm is incident normally on a narrow single slit of width 0.2 mm. for a Fraunhoffer diffraction pattern, find the angular position of the first and the second maxima.

Ans. For single slit diffraction, the intensity function is given by

$$I = I_0 \frac{\sin^2 \alpha}{\alpha^2}$$

where $\alpha = \dfrac{\pi a \sin \theta}{\lambda}$ and θ is diffraction angle and a is slit width.

there $\quad \lambda = 500$ mm $= 5 \times 10^{-5}$ cm,

$\quad a = 0.02$ cm

Let θ_1 and θ_2 be the required angles. Let α_1 and α_2 be the corresponding vales of α.

The first two maxima occur at

$$\alpha_1 = 1.430\,\pi$$

and $$\alpha_2 = 2.462\,\pi$$

Now $$\alpha_1 = \frac{\pi a \sin\theta_1}{\lambda} \text{ and } \alpha_2 = \frac{\pi a \sin\theta_2}{\lambda}$$

$$\therefore \quad \frac{\pi a \sin\theta_1}{\lambda} = \alpha_1 = 1.430\,\pi \Rightarrow \sin\theta_1 = \frac{1.430\,\lambda}{a}$$

and $$\frac{\pi a \sin\theta_2}{\lambda} = \alpha_2 = 2.462\,\pi$$

$$\Rightarrow \quad \sin\theta_2 = \frac{2.462\,\lambda}{a}$$

$$\therefore \quad \theta_1 = \sin^{-1}\left(\frac{1.430 \times 5 \times 10^{-5}}{0.02}\right) = \sin^{-1}(0.0036)$$

$$\therefore \quad \theta_1 = 3.57 \times 10^{-3} \text{ radian}$$

$$\theta_2 = \sin^{-1}\left(\frac{2.462 \times 5 \times 10^{-5}}{0.02}\right)$$

or, $$\theta_2 = 6.15 \times 10^{-3} \text{ radian}$$

11. (a) What is Brewster's Law?
(b) How is Brewster's angle connected to the refractive index of the medium?
(c) Discuss the nature of polarization of the reflected and refracted rays?
(d) Describe a Nicol prism and discuss its use as a polarizer.

Ans. (a) Refer to Article 6.7 of Chapter 6.
(b) Refer to 6.7 of Chapter 6
(c) Refer to 6.7 of Chapter 6
(d) Refer to the Article 6.12 of Chapter 6.

LASER

Group B

Long-Answer Type Questions

1. (a) Define Einstein's A, B coefficients and deduce their mutual relation.
(b) Show that the ratio of spontaneous and stimulated emission is proportional to the cube of the frequency.
(c) Explain the construction and action of the optical resonator in ruby laser.

Ans. (a) Refer to Article 7.4 of Chapter 7.
(b) Refer to Article 7.4 of Chapter 7.
(c) Refer to the Article 7.7 of Chapter 7.

QUANTUM PHYSICS

Group A

Multiple-Choice Type Questions

1. (i) A stone is dropped from the top of a building. What happens to the de Broglie wavelength of the stone as it falls?

(a) It increases (b) It decreases

(c) Remains constant (d) The de Broglie wavelength cannot be defined.

Ans. (b)

(ii) Number of Oscillation modes for the electromagnetic standing waves of frequency to $\nu + \Delta\nu$ or the cavity radiation is proportional to

(a) ν (b) ν^2 (c) ν^4 (d) $\frac{h\nu}{e^{h\nu/(kT)} - 1}$

Ans. (b)

Group B

Long-Answer Type Questions

1. (a) An electron and proton has same de Broglie wave length. Prove that the energy of electron is greater.

(b) Why Compton effect cannot be observed with visible light?

(c) Show that the temperature dependence in Stephan's law can be derived from Planck's radiation law.

Ans. (a) Refer to worked out problem 9.20 of Chapter 9.

(b) The wave length of the visible light is $\lambda = 6000$ Å.

So, energy of the visible light $E_\nu = \frac{hc}{\lambda}$

or, $$E_\nu = \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{6000 \times 10^{-10} \times 1.6 \times 10^{-19}} eV$$

or, $$E_\nu \approx 2\ eV$$

But the binding energy of an electron in the atom is $E_b \cong 10\ eV$.

So, when visible light falls on a target, it cannot liberate electrons of the scatterer. So we cannot observe Compton effect with visible light.

(c) Refer to Article 9.2.6 (b).

2. (a) Derive the expression for Compton shift when a photon interacts with electron. Show that it is impossible to transfer all the energy of the photon to the electron.

(b) Describe an experiment which verifies de Broglie's hypothesis about the wave nature of particles.

(c) Show that the product of group velocity and phase velocity of the de Broglie wave are constant.

Ans. For the first part refer to Article 9.4. Answer of the second part is as follows:

Let us assume that a photon transfer all its energy to free electron at rest. Now from the law of conservation of energy

$$h\nu + m_0 c^2 = [m_0^2 c^4 + p^2c^2]^{\frac{1}{2}} \quad ...(1)$$

where $m_0 c^2$ is the rest mass energy of the electron and $h\nu$ is the initial energy of the photon and from the law of conservation of momentum, we have

$$\frac{h\nu}{c} + 0 = 0 + p$$

or, $$h\nu = pc \quad ...(2)$$

So, from equations (1) and (2), we have

$$pc + m_0 c^2 = [m_0^2 c^4 + p^2c^2]^{\frac{1}{2}}$$

or, $$m_0^2 c^4 + p^2c^2 = p^2c^2 + 2m_0 pc^3 + m_0^2 c^4$$

or, $$2m_0 pc^3 = 0$$

or, $$2h\nu\, m_0 c^2 = 0 \quad [\because \quad h\nu = pc]$$

$\therefore$ $$\text{either } h\nu = 0 \quad \text{or} \quad m_0 c^2 = 0$$

which is impossible so, it is impossible for a photon to give all its energy to a free photon to give all its energy to a free electron.

(b) Refer to Article 9.5.8 of Chapter 9.

(c) Refer to Article 9.5.4 of Chapter 9.

Solved WBUT Questions of 2009

OSCILLATIONS (SHM, FDV and FV)

Group A

Multiple-Choice Type Questions

1 (i) Superposition of two SHM waves of equal time period and equal amplitude with phase difference $\phi = \frac{\pi}{2}$ forms a

(a) circle (b) ellipse (c) parabola

Ans. (a)

(ii) If α is the force constant of an oscillating body of mass m, the Q-factor is given by

(a) $Q = C\sqrt{\alpha m}$ (b) $Q = C\sqrt{m/\alpha}$ (c) $Q = C\sqrt{\alpha/m}$

Ans. (b)

(iii) When a spring with spring constant k is cut into three equal parts, the force constant of each of the part would be

(a) $k/3$ (b) $3k$ (c) k

Ans. (b)

Group B

Long-Answer Type Questions

1. (a) Show that $y = ae^{i(\omega t - kx)}$ is solution of wave equation.

(b) The potential energy of a particle with mass 10 g is given by, $V(x) = 32\,x^2 + 0.2$, where x is in metre and V is in joule. Write down the equation of motion and solve it.

Ans. (a) The wave equation is given by

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

The given function is

$$y = ae^{i(\omega t - kx)} \qquad ...(2)$$

Differentiating it with respect to x and t twice we get,

$$\frac{\partial y}{\partial t} = +\, i\omega ae^{i(\omega t - kx)}$$

or, $$\frac{\partial^2 y}{\partial t^2} = -\,\omega^2 ae^{i(\omega t - kx)}$$

or, $$\frac{\partial^2 y}{\partial t^2} = -\,\omega^2 y \qquad ...(3)$$

Again, $$\frac{\partial y}{\partial x} = -\,ikae^{i(\omega t - kx)}$$

or, $$\frac{\partial y}{\partial x^2} = (-\,ik)^2\, ae^{i(\omega t - kx)}$$

or, $$\frac{\partial^2 y}{\partial x^2} = -\,k^2 y \qquad ...(4)$$

From Eqs. (3) and (4), we can write

$$\frac{1}{\omega^2}\frac{\partial^2 y}{\partial t^2} = \frac{1}{k^2}\frac{\partial^2 y}{\partial x^2}$$

or, $$\frac{\partial^2 y}{\partial t^2} = \left(\frac{\omega}{k}\right)^2 \frac{\partial^2 y}{\partial x^2}$$

or, $$\frac{\partial^2 y}{\partial t^2} = \frac{(2\pi\nu)^2}{\left(\frac{2\pi}{\lambda}\right)^2} \cdot \frac{\partial^2 y}{\partial x^2}$$

or, $$\frac{\partial^2 y}{\partial t^2} = (\nu\lambda)^2 \frac{\partial^2 y}{\partial x^2}$$

or, $$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \quad ...(5)$$

where $v = \nu\lambda$

Equations (1) and (5) are of the same form.

$\therefore$ $y = ae^{i(kx)}$ represents a wave equation.

(b) Here mass $m = 10\text{ g} = 10^{-2}\text{ kg}$

The potential energy is given by

$$V(x) = 32\,x^2 + 0.2 \quad ...(1)$$

$\therefore$ Force $F(x) = -\dfrac{dV(x)}{dx}$

or, $F(x) = -(64\,x)$

or, $F(x) = -64\,x$

or, $F(x) = -kx$ where $k = 64$ J/m

or, $$m\frac{d^2x}{dt^2} = -kx$$

or, $$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

or, $$\frac{d^2x}{dt^2} + \frac{64}{10}x = 0$$

or, $$\frac{d^2x}{dt^2} + 6.4\,x = 0 \quad ...(2)$$

The Eq. (2) is the required equation.

For the answer of the second part of the equation refer to Article 1.5 of Chapter 1.

2. (a) Establish the differential equation of damped harmonic motion.

(b) Solve the equation for light damping and prove that the amplitude of vibration decreases exponentially with time.

(c) A cubical block of side L cm and density d is floating in a water of density $\rho(\rho > d)$. The block is slightly depressed and then released. Show that it will execute simple harmonic motion and hence determine the frequency of oscillation.

Ans. (a) Refer to Article 2.3 of Chapter 2.

(b) Refer to Article 2.4 of Chapter 2.

(c)

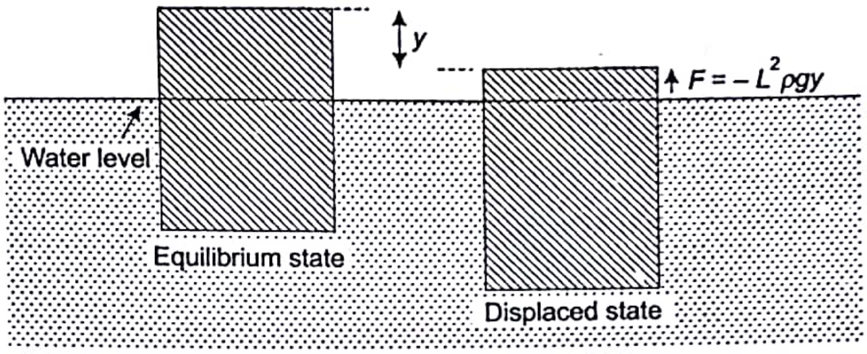


Fig. 1 Oscillation of a floating black

Let m be the mass of the block. If the block is displaced a distance y below its equilibrium position, then the vuoyant force increases by $(L^2y)\ \rho g$ because L^2y is the additional volume of liquid displaced hence $L^2y\ \rho$ is the mass and $L^2y\ \rho g$ is the weight of liquid displaced further.

Thus the restoring force is given by

$$F = -(L^2 \rho g)\ y = -ky$$

Since the restoring force is proportional to the displacement, the motion is SHM of angular frequency

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{L^2 \rho g}{m}}$$

or, $$\omega = \sqrt{\frac{(L^3 P)g}{Lm}} = \sqrt{\frac{mg}{Lm}} \qquad [\because \ m = L^3 \rho]$$

or, $$2\pi\nu = \sqrt{\frac{g}{L}}$$

$\therefore$ $$\nu = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

which is the required frequency.

OPTICS (Interference, Diffraction and Polarization)

Group A

Multiple-Choice Type Questions

1. (i) Newton's ring experiment is based on
   (a) division of amplitude (b) division of wave front
   (c) none of these.

Ans. (a)

(ii) In a plane transmission grating light
   (a) diffracts to produce the resultant pattern

(b) diffracts and interferes to produce the resultant pattern

(c) interferes to produce the resultant pattern.

Ans. (c)

Group B

Long-Answer Type Questions

1. (a) What is the difference between temporal coherence and spatial coherence

(b) If the amplitudes of two coherent light waves are in the ratio 1 : 4, find the ratio of maximum and minimum intensity in the interference pattern.

Ans. (a) The temporal coherence of any wave field implies the possibility of predicting phase and amplitude at a point in space at different instants of time. A monochromatic wave field is temporally coherent. The spatial coherence is concerned with phase correlation between two wave fields at two space points at the same instant of time.

Let a_1 and a_2 be amplitudes of two coherent light waves.

$\therefore \quad a_1 : a_2 = 1 : 4$

or, $\quad a_2 = 4a_1$

$\therefore$ the ratio of the maximum to minimum intensity is given by

$$\frac{I_{\max}}{I_{\min}} = \frac{(a_1 + a_2)^2}{(a_2 \sim a_1)^2} = \left(\frac{5a_1}{3a_1}\right)^2 = \frac{25}{9}$$

i.e., $I_{\max} : I_{\min} = 25 : 9$.

2. (a) What is double refracting crystal?

(b) Discuss Nicol prism as polarizer and analyzer.

(c) Determine the Brewster's angle for glass of refractive index 1.5 immersed in water of refractive index 1.33.

(d) Prove that the intensity of secondary maxima formed for Fraunhofer diffraction at a single slit are of decreasing order.

(e) In a plane transmission grating the angle of diffraction for second order maxima for wave length 5×10^5 cm is 30°. Calculate the number of lines in one centimeter of the grating surface.

Ans. (a) It is a transparent crystal which can generate two refracted beams of light when an ordinary light ray is incident on it. One of the refracted rays obeys the laws of refraction and is called ordinary ray and the other ray does not obey the laws of refraction and is called the extra-ordinary ray. Calcite crystal is an example of doubly refracting crystal.

(b) Refer to Article 6.12.2 of Chapter 6.

(c) The refractive index of glass is given by ${}_a\mu_g = 1.5$ and the refractive index of water is given by ${}_a\mu_w = 1.33$

$\therefore$ The refractive index of glass with respect to water is given by

$${}_\omega\mu_g = \frac{{}_a\mu_g}{{}_a\mu_\omega} = \frac{1.50}{1.33} = 1.128$$

If θ_P be the angle of polarization in water, then

$$_{\omega}\mu_g = \tan\theta_P$$

or, $$\tan\theta_P = 1.128$$

$\therefore$ $$\theta_P = \tan^{-1}(1.128) = 48.4°$$

LASER

Group A

Multiple-Choice Type Questions

1. (i) In the He–Ne laser, the laser light emits due to the transition from

(a) $3s \to 2p$ (b) $3s \to 3p$ (c) $2s \to 2p$

Ans. (a)

Group B

Long-Answer Type Questions

1. (a) Describe briefly the working principle of laser action.

Ans. (a) Refer to Article 7.5 of Chapter 7.

2. (a) In a He-Ne laser transition from $3s$ to $2p$ level gives a laser beam of wave-length 632.8 nm. If the $2p$ level has energy equal to 15.2×10^{-19} J. Calculate the required pumping energy (assuming no loss of energy).

Ans. The pumping takes place between the energy levels E_0 and E_{3s} and the transition takes place between E_{3s} and E_{2p} levels.

$\therefore$ $$\Delta E = E_{3s} - E_{2p} = \frac{hc}{\lambda}$$

or, $$\Delta E = \frac{6.627 \times 10^{-34} \times 3 \times 10^{8}}{632.8 \times 10^{-9}} \text{ J}$$

or, $$\Delta E = 3.14 \times 10^{-19} \text{ J}$$

The required pumping energy is given by

$$E = E_{3s} - E_0 = E_{2p} + \Delta E$$

or, $$E = 15.20 \times 10^{-19} + 3.14 \times 10^{-19} \text{ J}$$

$\therefore$ $$E = 18.34 \text{ J}$$

3. (a) Discuss the operation of ruby laser with the help of energy level diagram.

(b) What is the role of optical resonator in laser production?

Ans. (a) Refer to Article 7.10.1 of Chapter 7.

(b) Refer to Article 7.8 of Chapter 7.

QUANTUM PHYSICS

Group A

Multiple-Choice Type Questions

1. (i) The de Broglie wave length of a particle of mass m and kinetic energy E is

(a) $\lambda = \dfrac{h}{2mE}$ (b) $\dfrac{h}{\sqrt{2mE}}$ (c) $\sqrt{\dfrac{2mE}{h}}$

Ans. (b)

(ii) Mass of a photon of frequencies ν is given by

(a) $\dfrac{h\nu}{c}$ (b) $\dfrac{h\nu}{c^2}$ (c) $\dfrac{h\nu^2}{c}$

Ans. (b)

Group B

Long-Answer Type Questions

1. (a) What is Compton effect? Calculate the Compton wave length for an electron.

(b) Why does the unmodified line appear in Compton scattering?

Ans. (a) It is a phenomenon of collision between a photon and a loosely bound electron of an atom. When a photon of energy $h\nu$ collides with a free (or loosely bound) electron of the scatterer at rest, it transfers some of its energy to the electron. The scattered photon has a smaller energy $h\nu'$ ($< h\nu$) and consequently a greater wavelength than that of the incident photon. The change in wave length of the scattered photon by considering the elastic collision between the incident photon and the free electron is known Compton shift.

The Compton wavelength of an electron is given by

$$\lambda_c = \frac{h}{m_0 c}$$

where m_0 is the rest mass of the electron.

$$\therefore \quad \lambda_c = \frac{6.626 \times 10^{-34}}{9.1 \times 10^{-31} \times 3 \times 10^{8}} = 2.426 \times 10^{-12}\ \text{m}$$

or, $\lambda_c = 0.02426$ Å

When an x-ray photon collides with a free electron (or very loosely bound electron whose work-function is less). The wave length shift of the photon is given by

$$\Delta\lambda = \frac{h}{m_0 c}(1 - \cos\phi)$$

where m_0 is the rest mass of the scattered electron. As the mass of the electron is very much small, $\Delta\lambda$ is appreciably high, and a modified line is observed. On the other hand, when the x-ray photon collides with a tightly bound electron, the photon cannot strip off the electron from the atom, the collision becomes that of the atom and the photon.

In this case m_0 represents the mass of the atom. So, the wavelength shift $\Delta\lambda$ becomes almost zero. For this reason, unmodifield lines are observed.

2. (a) State and explain de Broglie hypothesis.
   (b) Prove that the product of phase velocity and group velocity for a de Broglie wave is equal to the square of the light velocity.
   (c) Compute the smallest possible uncertainly in the position of an electron moving with velocity 3×10^7 m/s. The rest mass of electron is 9.1×10^{-31} kg.
   (d) Derive Wein's displacement law from Planck's radiation law.

Ans. (a) Refer to Article 9.5.1 of the Chapter 9.
(b) Refer to Article 4.5.4
(c) The uncertainly principle in terms of position and momentum is given by

$$\Delta x \, \Delta p_x \geq \hbar$$

If $(\Delta x)_{\min}$ be the smallest possible uncertainly in position, then

$$(\Delta x)_{\min} \, (\Delta p_x)_c = \hbar$$

where $(\Delta p_x)_c$ is the corresponding uncertainly in momentum

Here, velocity of the electron is

$$v_e = 3 \times 10^7 \text{ ms}^{-1}$$

and mass of the electron is

$$m_e = 9.1 \times 10^{-31} \text{ kg}$$

So $\quad (\Delta p_x)_c = 9.1 \times 10^{-31} \text{ kg} \times 3 \times 10^7 \text{ ms}^{-1}$

or, $\quad (\Delta p_x)_c = 27.3 \times 10^{-24} \text{ kg ms}^{-1}$

$\therefore \quad (\Delta x)_{\min} = \dfrac{\hbar}{(\Delta P_x)_c} = \dfrac{1.055 \times 10^{-34} \text{ Js}}{27.3 \times 10^{-24} \text{ kg ms}^{-1}}$

or $\quad (\Delta x)_{\min} = 0.0386 \times 10^{-10}$ m

$\therefore \quad (\Delta x)_{\min} = 3.86 \times 10^{-12}$ m

Refer to Article 9.26 of Chapter 9.

3. Why Compton effect cannot be observed with visible light but can be observed due to x-rays?

Ans. The Compton effect cannot be observed with visible light because the energy possessed by a photon of visible light is not sufficient to cause the Compton effect. Let us consider a visible light photon having wave length $\lambda = 600$ mm, i.e, $\lambda = 600 \times 10^{-9}$ m

The energy possessed by the photon is

$$E = \frac{hc}{\lambda} = \frac{6.627 \times 10^{-31} \times 3.0 \times 10^8}{600 \times 10^{-9} \times 1.6 \times 10^{-19}} \text{ eV}$$

or, $\quad E \approx 2$ eV.

The binding energy of an electron in the atom is $E_B \cong 10$ eV. As for example the binding energy of an electron of a hydrogen atom is 13.6 eV.

So when a photon of visible light falls on a target, it fails to liberate electrons. So, one cannot see Compton effect in case of a visible light.

But an x-ray photon can cause Compton effect as its energy is far more than 10 eV.

CRYSTALLOGRAPHY

Group A

Multiple-Choice Type Questions

1. (i) Miller indices of a plane which cut intercepts of 2, 3 and 4 units along the three axes are

(a) (2, 3, 2) (b) (2, 3, 4) (c) (6, 4, 3)

Ans. (c).

(ii) The atomic radius of a face centred cubic crystal of lattice constant a is

(a) $\frac{a}{2}$ (b) $\frac{\sqrt{3}a}{4}$ (c) $\frac{\sqrt{2}a}{4}$

Ans. (c)

(iii) An x-ray tube is subjected to a potential difference of 50 kV with the corresponding current of 8 mA through it. The number of electrons striking per second on the target material is

(a) 5×10^{16} (b) 6×10^{11} (c) none of these

Ans. (a)

Group B

Long-Answer Type Questions

1. (a) Deduce the formulae for interplaner spacing of a simple cubic crystal.

(b) Why x-ray diffraction is used for crystal structure analysis?

Ans. (a) Refer to Article 10.9 of Chapter 10.

(b) Refer to the answer of question No. 7 of "Short questions with Answer" under 'Review exercise' of Chapter 10.

Solved WBUT Questions of 2010-11

Group A

Multiple-Choice Type Questions

1. Choose the correct alternatives for any *ten* of the following: $10 \times 1 = 10$

(i) If a particle is executing simple harmonic motion with frequency v then its potential energy

(a) remains constant over time (b) is oscillating with a frequency v

(c) is oscillating with a frequency $v/2$ (d) is oscillating with a frequency $2v$

(ii) The quality factor Q for an L-C-R circuit is

(a) $\frac{\omega R}{L}$ (b) $\frac{\omega L}{R}$ (c) $\frac{\omega}{LR}$ (d) $\frac{R}{\omega L}$

(iii) An external force $F = F_0\, e^{i\omega x}$ is applied to a slightly damped oscillator of natural frequency ω_0, then in steady state it will oscillate with a frequency

(a) ω_0 (b) ω (c) $\omega_0 - \omega$ (d) $\sqrt{\omega_2^0 - \omega^2}$

(iv) The intensity of principal maximum in the Fraunhofer diffraction spectrum produced by a grating with N number of lines is proportional to

(a) $\frac{1}{N}$ (b) N (c) N^2 (d) $\frac{1}{N^2}$

(v) In the propagation of lightwave the angle between the plan of polarization and plane of vibration is

(a) 0° (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) π

(vi) In case of simple cubic crystal, the effective number of atoms per unit cell is

(a) 2 (b) 1 (c) 3 (d) none of these

(vii) In order to produces X-ray by striking electron beam on a metal target the kinetic energy of the electron should be

(a) less than one eV (b) of the order of eV
(c) of the order of keV (d) of the order of GeV

(viii) If the energy of a particle is much higher than its rest energy then the energy is

(a) independent of the momentum
(b) proportional of the momentum
(c) proportional to square of the momentum
(d) proportional to the square root of the momentum

(ix) If a wave packet is described by

$\varphi(x) = A\, exp\left(-\frac{x^2}{2\sigma^2}\right)$ then the momentum uncertainty is proportional to

(a) $h\sigma$ (b) $\frac{h}{\sigma}$ (c) $h\sigma^2$ (d) $\frac{h}{\sigma^2}$

(x) Given that the temperature of a younger star is higher than that of an older one,

(a) a blue star is younger than a red star
(b) a blue star is older than a red star
(c) both of them are same age
(d) colour of the star cannot be correlated to the age of the star

(xi) In ruby laser, the active medium is

(a) solid (b) liquid
(c) gas (d) a solid and gas mixture

(xii) In a Nicol-prism the O-ray is totally internally reflected and the E-ray is transmitted. This statement is

(a) true (b) false (c) partly true (d) partly false

(xiii) Two sources are said to be coherent when the waves produced by them have
- (a) same wavelength
- (b) same wavelength and same phase
- (c) same wavelength and constant phase difference
- (d) same amplitude and constant phase difference

(xiv) In an arrangement for viewing Newton's ring, if the lens which rests on a glass plate were moved upwards by one wavelength, (of the viewing light), which of the following will be observed?
- (a) The central spot becomes bright.
- (b) No change of fringe pattern is observed.
- (c) The rings shift towards the centre.
- (d) The rings move out from the centre.

(xv) Holography is base on
- (a) the interference of reference and object waves which are coherent
- (b) superposition of object and the reference waves which are of slightly different wavelengths.
- (c) recording of superposed images of two different wavelengths
- (d) recording the phase information of the resultant of the reference and the object waves

(xvi) For a laser action to occur, the medium used must have at least
- (a) 4 energy levels
- (b) 2 energy levels
- (c) 3 energy levels
- (d) one energy level

Group B

Short-Answer-Type Questions

Answer any *three* of the following. $3 \times 5 = 15$

2. A vibrator of 10 g mass is acted on by a restoring force of 5 dyne/cm and a damping force 2 dyne-s/cm. Find whether the motion is overdamped or oscillatory. If at $t = 0$ the vibrator was at position $x = 0$, when a velocity 1 cm/sec is imparted to it then calculate the maximum deviation along positive x-axis. 2 + 3

Ans. Here, mass of the oscillator $m = 10$ g,
restoring force $k = 5$ dyne/cm and
damping force $\beta = 2$ dyne-s/cm
The equation of the damped oscillator is given by

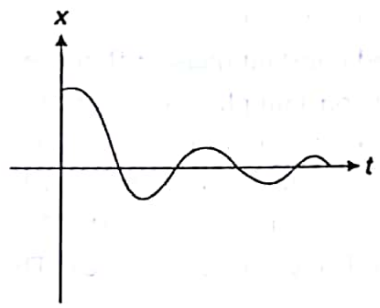
$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0$$

While solving this equation, we get damping constant $b = \dfrac{\beta}{2m} = \dfrac{2}{2 \times 10} = 0.1$ and the angular frequency $\omega = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{5}{10}} = 0.705$

In this case, $b < \omega$, so the motion is oscillatory.
Now, since the motion is damping and oscillatory, the amplitude of oscillation will decrease exponentially as follow:

$$A(t) = A_0\, e^{-bt}$$

where A_0 is the expected constant amplitude in the absence of damping. The vibration of the displacement will be as given in the following figure:



As can be seen the amplitude will be maximum when $t = \frac{T}{4}$ where T is the time period of oscillation.

$$\therefore \qquad A_{max} = A\left(\frac{T}{4}\right) = A_0\, e^{-bt/4}$$

we know that $\quad T = \dfrac{2\pi}{\sqrt{\omega^2 - b^2}}$

$$\therefore \qquad T = \frac{2\pi}{\sqrt{0.50 - 0.01}} = \frac{2\pi}{\sqrt{0.49}} = \frac{2\pi}{0.7}$$

$$\therefore \qquad A_{max} = A_0\, e^{-0.1 \times \frac{2\pi}{0.7}/4}$$

$$\text{or,} \qquad A_{max} = A_0\, e^{-\frac{2\pi}{28}}$$

$$\therefore \qquad A_{max} = A_0\, e^{-\pi/14} \text{ unit.}$$

3. (a) In a Newton's ring experiment, the diameter of a dark ring is 0.32 cm, when the wavelength of monochromatic light be 6000 Å. What would be the diameter of that ring when the wavelength of light changes to 5000 Å? 2

Ans. The diameter of the dark ring is given by

$$D_n^2 = 4n\lambda R$$

$$\text{or,} \qquad 4nR = \frac{D_n^2}{\lambda} = \frac{0.32 \times 0.32}{6000 \times 10^{-8}} = \frac{22 \times 32}{6} \times 10^{-9}$$

When wavelength $\lambda' = 5000$ Å, the diameter is

$$D_n'^2 = 4nR\lambda' = \frac{32 \times 32}{6} \times 10^{-9} \times 5000 \times 10^{-8}$$

$$= \frac{32 \times 32 \times 5}{6} \times 10^{-4}$$

$$\text{or,} \qquad D_n' = 29.21 \times 10^{-2} \text{ cm}$$

$$\therefore \qquad D_n' = 0.29 \text{ cm}$$

(b) Write down the expression for the intensity of light due to Fraunhoffer diffraction in a transmission grating and hence find the condition for secondary minima in the interference pattern. [See Articles 5.7] 1 + 2

4. (a) Two polarizers are placed at crossed position and angle between the polarizing planes are 90°. A third polarizer with angle θ with the first one is placed between them. An unpolarized light of intensity I is incident on the first one and passes through all three polarizers. Find the intensity of the light that comes out. [Worked out Problems Problem 6.3] 3

(b) What is population inversion in the context of LASER? Does it violate Maxwell–Boltzmann distribution law? [See Article 7.6; No, it does not violate] 2

5. (a) An electron is observed moving at 50% of the speed of light, $v = 1.5 \times 10^8$ m/s. What is the relativistic mass of the electron? What is the kinetic energy of the electron? 2

Ans. The relation between relativistic mass and rest mass is given by

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

Here, $v = 1.5 \times 10^8$ m/s and $c = 3 \times 10^8$ m/s

$$\therefore \quad m = \frac{m_0}{\sqrt{1 - \dfrac{1.5 \times 1.5 \times 10^{16}}{3 \times 3 \times 10^{16}}}}$$

$$\text{or,} \quad m = \frac{2}{\sqrt{3}} m_0$$

The relativistic mass of the electron is $\frac{2}{\sqrt{3}} m_0$ the kinetic energy (T) of the electron is

$$T = mc^2 - m_0c^2$$

$$= \frac{2}{\sqrt{3}} m_0c^2 - m_0c^2$$

$$\text{or,} \quad T = \left(\frac{2 - \sqrt{3}}{\sqrt{3}}\right) m_0c^2 = 0.15\, m_0c^2$$

(b) Write down the conservation laws in Compton scattering. Show that in Compton scattering while the photon can be scattered at any angle between 0° to 180°, the recoil electron can only be emitted at angles between 0° and 90°. [See Article 9.4.1 and 9.4.4] 2 + 1

6. (a) The linear absorption coefficient of an element of X-rays having $\lambda = 0.3$Å is 135 m^{-1}. Find the half value thickness for that material. 2

Ans. Here $\lambda = 0.3$ Å

If μ be the linear absorption coefficient them intensity $I = I_0 e^{-\mu x}$ where I_0 and I are intensities of x-ray before and after passes through the thickness x.

Here $I = \frac{I_0}{2}$, so, $x = \frac{0.693}{\mu} = \frac{0.693}{135}$ m

so, half-value thickness of the material is 0.0051 m

(b) Cs metal (atomic weight 130) has a cubic unit cell of lattice constant 0.6 nm. If the density of Cs is 2 g/cc, determine whether the unit cell is simple, body-centered or face-centered. 3

Ans. For a cubic crystal, the lattice constant

$$a = \left(\frac{n M_A}{\rho N}\right)^{1/3}$$

Here n is the number of atom per unit cell. Atomic weight $M_A = 130$, density of Cs $\rho = 2$ g/cc

Lattice constant $a = 0.6$ nm

$$= 0.6 \times 10^{-9} \text{ m}$$

$$= 0.6 \times 10^{-7} \text{ cm}$$

and $N = 6.023 \times 10^{23}$/g mole

$$\therefore \quad n = \frac{a^3 \rho N}{M_A} = \frac{(0.6 \times 10^{-7})^3 \times 2 \times 6.023 \times 10^{23}}{130}$$

$$= \frac{0.6 \times 0.6 \times 0.6 \times 10^{-21} \times 2 \times 6.023 \times 10^{23}}{130}$$

$$\therefore \quad n = 2$$

The unit cell is body centred.

Group C

Long-Answer-Type Questions

Answer any *three* of the following. $3 \times 15 = 45$

7. (a) Starting from the equation of motion and after solving it show that, for a forced oscillator in the steady state, the displacement amplitude at low frequencies ($\omega \to 0$), the velocity amplitude at velocity resonance ($\omega = \omega_0$) are independent of the frequency of the driving force.

$2 + 6 + 2 + 2$

Ans. For developing and solving the differential equation of a forced oscillator refers to the Article 3.3. The complete solution of the said differential equation is given by

$$y = pe^{-bt} \cos \{(\sqrt{\omega^2 - b^2})\, t - \theta\}$$

$$+ \frac{f}{\sqrt{\{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}} \cos \left\{\omega' t - \tan^{-1}\left(\frac{2b\omega'}{\omega^2 - \omega'^2}\right)\right\}$$

where ω' is the regular frequency of the driving force.

At steady state condition the first part of the equation is zero.

$\therefore$ The equation then given by

$$y = A \cos (\omega' t - \alpha)$$

where
$$A = \frac{f}{\sqrt{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}}$$

and
$$\alpha = \tan^{-1}\left(\frac{2b\omega'}{\omega^2 - \omega'^2}\right)$$

So, velocity $v = \frac{dy}{dt} = -A\omega' \sin(\omega' - \alpha)$

Hence, the velocity amplitude is given by

$$A_v = A\omega'$$

or, $$A_v = \frac{f\omega'}{\sqrt{(\omega^2 - \omega'^2) + 4b^2\omega'^2}}$$

or, $$A_v = \frac{f}{\sqrt{\frac{(\omega^2 - \omega'^2)^2 + 4b^2\omega'^2}{\omega'}}}$$

or, $$A_v = \frac{f}{\sqrt{\left(\frac{\omega^2}{\omega'} - \omega'\right)^2 + 4b^2}}$$

or, $$A_v = \frac{f}{\sqrt{\omega^2 \Delta^2 + 4b^2}}$$

where $$\Delta = \frac{\omega}{\omega'} - \frac{\omega'}{\omega}$$

At velocity resonance, $\omega = \omega'$

$\therefore$ $$\Delta = 0$$

$\therefore$ $$A_v = \frac{f}{4b^2}$$

Hence, the velocity amplitude A_v is independent of ω', i.e., the frequency of the driving force.

(b) Explain the terms logarithmic decrement and quality factor of a damped oscillatory system. How are they related? 1 + 1 + 1

Ans. Refer to Article 2.4 for logarithmic decrement.

In case of damped oscillator the amplitude of vibration is time dependent and it is given by

$$A(t) = A_0 e^{-bt}$$

where $$A_0 = \frac{\omega y_0}{\sqrt{\omega^2 - b^2}}$$

As can be seen in the equation $A(t) = A_0 e^{-bt}$. The amplitude decreases exponentially, for this reason, it is called logarithmic decrement.

For explanation of the quality factor, refer to Article 3.7.

When an external force is applied to a damped oscillator, it obtains a steady state oscillation and that time there does not remain any logarithmic decrement of the amplitude. And only this time we can obtain an expression for the quality factor $Q\left(=\frac{\omega}{2b}\right)$.

So, the quality factor and logarithmic decrement are not functionally dependent.

8. (a) Derive the intensity distribution of diffraction of Fraunhofer class of light due to a single slit. Sketch the intensity distribution. [See Article 5.4] 7 + 2

(b) A single slit forms a diffraction pattern of Fraunhofer class with white light. The second maximum in the pattern for red light of wavelength 7000 Å coincides with the third maximum of an unknown wavelength. Calculate the unknown wavelength. 3

Ans. Here $n_1\lambda_1 = n_2\lambda_2$

where n_1 and n_2 are order members $n_1 = 2$, $\lambda_1 = 7000$ Å

$$\lambda_2 = \frac{n_1\lambda_1}{n_2} = \frac{2 \times 7000}{3} \text{ Å} = \frac{14}{3} \times 1000 \text{ Å}$$

$\therefore$ $\lambda_2 = 4667$ Å

(c) In Young's double-slit experiment the distance between two slits is 0.5 mm. The wavelength of light is 5000 Å and the separation between the sources and the screen is 50 cm. Calculate the Fringe width in this case. 3

Ans. The fringe width $\beta = \dfrac{D\lambda}{2d}$

Here, $2d = 0.5 \text{ mm} = 0.05 \text{ cm}$,

$\lambda = 5000 \text{ Å} = 5000 \times 10^{-8} \text{ cm}$

and $D = 50$ cm

so, $\beta = \dfrac{50 \times 5000 \times 10^{-8}}{0.05}$ cm

$\therefore$ $\beta = 0.05$ cm

9. (a) How can you get an elliptically polarized light from an linearly polarized light by using an optical device?

A plane polarized light is incident on a piece of quartz cut parallel to the axis. Find the least thickness for which the ordinary and the extraordinary rays comine to form plane polarized light given that $\mu_0 = 1.5442$ and $\mu_e = 1.5533$, $\lambda = 5 \times 10^{-5}$ cm. 2 + 3

Ans.

$$t = \frac{\lambda}{4\,(\mu_e - \mu_0)} = \frac{5 \times 10^{-5}}{4 \times (1.5533 - 1.5442)} \text{ cm}$$

$$= \frac{5 \times 10^{-5}}{4 \times 0.0091} \text{ cm} = \frac{5}{4 \times 91} \times 10^{-1} \text{ cm}$$

$\therefore$ $t = 0.0013$ cm

(b) Define Einstein *A*, *B* coefficients of absorption and emission. Find out the relations among them. [see Article 7.4] 1 + 1 + 1 + 3

(c) Explain how holographic images are reconstructed from the holograms. [See Articles 8.3] 4

10. (a) Describe the function of an optical resonator. What is the use of such a device in the context of LASER generation? [See Article 7.8] 3 + 1

(b) What is coordination number? Determine the coordination number for *sc* and *bcc* structures. Draw necessary sketch, 'atomic packing factor' increases with co-ordination number. Justify the statement. 1 + 4 + 3

Ans. Refer to the Article 7.4.

The atomic packing factor (APF) can be expressed by the following mathematical formula

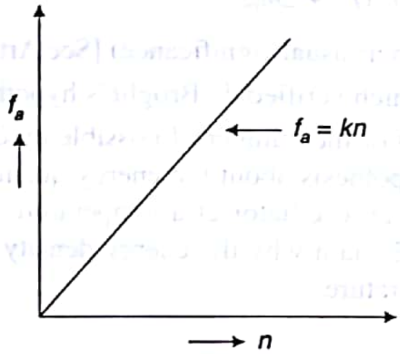
$$f_a = \frac{n\,v}{V}$$

where f_a is the atomic packing factor and V is the volume of a unit cell of the crystal and v in the volume of each atom $\left(v = \frac{4}{3}\pi r^3\right)$ and n is the coordination mumber. In the above equation, v and V are constants for a particular type of matter (i.e., element). So, we see that

$$f_a \propto n$$

$$\therefore \quad f_a = \frac{v}{V}n = kn \text{ where } k = \frac{v}{V} = \text{constant}$$

If we now plot f_a against n we get the following curve:



(c) Find the short-wave limit of continuous X-ray spectrum. Does it shift by 0.5 nm when the voltage applied to X-ray tube is doubled? 2 + 1

Ans. Duane Hunt law gives us minimum wave-length (i.e., short wave limit) of continuous X-ray as follows:

$$\lambda_{\min} = \frac{hc}{eV}$$

or, $$\lambda_{\min} = \frac{6.62 \times 10^{-34} \times 3 \times 10^{8}}{1.6 \times 10^{-19}\, V}\text{ m}$$

or, $$\lambda_{\min} = \frac{12.4125}{V} \times 10^{-7}\text{ m} \quad \text{(if } V \text{ is in volt)}$$

or, $$\lambda_{\min} = \frac{12412.5}{V}\text{ Å} = \frac{1241.25}{V}\text{ nm}$$

If the voltage V is doubled then λ' min will be given by

$$\lambda_{\min} = \frac{1241.25}{V_1} = \frac{1241.25}{2V}\text{ nm}$$

where $V_1 = 2V$

when voltage is doubled, the wave length shift is given by

$$\Delta\lambda = \lambda_{\min} = -\lambda'_{\min}$$

or, $$\Delta\lambda = \frac{1241.25}{V} - \frac{1241.25}{2V}$$

or, $$\Delta\lambda = \frac{1241.25}{V}\left(1 - \frac{1}{2}\right)$$

or, $$\Delta\lambda = \frac{620.625}{V} \text{ nm}$$

so, if V be 1241.25 volt only then wave-length shift will be 0.5 nm.

11. (a) State and explain de Broglie's hypothesis. Show that the relativistic de Broglie wavelength is given by

$$\lambda_{relativistic} = \frac{h}{\sqrt{E_k(E_k + 2m_0c^2)}}$$

(The notations used have their usual significance) [See Article 9.5.6] 2 + 3

(b) Describe an experiment which verified de Broglie's hypothesis. [See Article 9.5.8] 4

(c) Write down the expression of the number of possible modes of cavity waves of frequency ν to $\nu + d\nu$. Using Planck's hypothesis about the energy quantization of the cavity oscillators, find out the average energy of an oscillator at a temperature T and the energy density within the frequency range ν to $d\nu$. Explain why the energy density decreases at very high values of the frequency at a finite temperature. 1 + 3 + 1 + 1

[See Article 9.2.3 & 9.2.5]

5744